GUDLAVALLERU ENGINEERING COLLEGE

(An Autonomous Institute with Permanent Affiliation to JNTUK, Kakinada)

Seshadri Rao Knowledge Village, Gudlavalleru – 521 356.

Department of Electronics and Communication Engineering

HANDOUT

on

PROBABILITY THEORY AND RANDOM VARIABLES

Vision

To be a leading centre of education and research in Electronics and Communication Engineering, making the students adaptable to changing technological and societal needs in a holistic learning environment.

Mission

- To produce knowledgeable and technologically competent engineers for providing services to the society.
- To have a collaboration with leading academic, industrial and research organizations for promoting research activities among faculty and students.
- To create an integrated learning environment for sustained growth in electronics and communication engineering and related areas.

Program Educational Objectives

Graduates of the Electronics and Communication Engineering program will

1. Demonstrate a progression in technical competence and leadership in the practice/field of engineering with professional ethics.
2. Continue to learn and adapt to evolving technologies for catering to the needs of the society.

HANDOUT ON PROBABILITY THEORY AND RANDOM VARIABLES

Class & Sem.	: II B.Tech – I Semester	Year	: 2017-18
Branch	: ECE	Credits	: 3

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1. **Brief History and Scope of the Subject**

   This is a graduate level course in probability theory and random variables that is a pre-requisite for almost all graduate level courses in communications, signal processing, controls and networks. The course will cover: axiomatic foundations of probability theory, random variables, distributions, densities, functions of a random variable, functions of several random variables, moment generating functions, random vectors, sequences, and functions, linear transformations, second moment theory, spectral densities, Gaussian processes, stationary processes, correlation detection and linear systems with random inputs.

2. **Pre-Requisites**
   - Fundamentals of Mathematics

3. **Course Objectives:**
   - To familiarize concepts of probabilityand random variables.
   - To imbibe students with some of the commonly encountered random variables, in particular the Gaussian random variable.
   - To acquaint students the moment generating and characteristic functions.
   - To make students conversant in classifications of random processes and concepts such as strictstationarity, wide-sense stationarity and ergodicity.
   - To indoctrinate in the concepts of correlation functions and power spectral density.

4. **Course Outcomes:**

At the end of the course, students will be able to:

- determine and understand probability, statistics of random variables and their functions.

- decide statistics of random vectors and their functions.
- calculate statistics of random sequences, random processes, and their input and output relationships and statistics in linear systems.
- apply the concepts of probability, random variables / processes to analyzestatistical problems in Electronics and communication Engineering field.
- apply the concepts of filtering and prediction of a random process

5. Program Outcomes:

The graduates of electronics and communication engineering program will be able to

a) apply knowledge of mathematics, science, and engineering for solving intricate engineering problems.
b) identify, formulate and analyze multifaceted engineering problems.
c) design a system, component, or process to meet desired needs within realistic constraints such as economic, environmental, social, political, ethical, health and safety, manufacturability, and sustainability.
d) design and conduct experiments based on complex engineering problems, as well as to analyze and interpret data.
e) use the techniques, skills, and modern engineering tools necessary for engineering practice.
f) understand the impact of engineering solutions in a global, economic and societal context.
g) design and develop eco-friendly systems, making optimal utilization of available natural resources.
h) understand professional ethics and responsibilities.
i) work as a member and leader in a team in multidisciplinary environment.
j) communicate effectively.
k) manage the projects keeping in view the economical and societal considerations.
l) recognize the need for adapting to technological changes and engage in life-long learning.

6. Mapping of Course Outcomes with Program Outcomes:

	a	b	c	d	e	f	g	h	i	j	k	l
CO1	H	H			M							L
CO2	H	H			M							L
CO3					H	M						L
CO4		M				H						L

7. Prescribed Text Books

Peebles Jr. P.Z., “Probability Random Variables and Random Signal Principles”, Tata McGraw-Hill Publishers, Fourth Edition, New Delhi, 2002.

8. Reference Text Books

a. Hwei Hsu, "Schaum's Outline of Theory and Problems of Probability, Random Variables and Random Processes", Tata McGraw-Hill edition, New Delhi, 2004.

b. H. Stark and J.W. Woods, "Probability and Random Processes with Applications to Signal Processing", Pearson Education (Asia), 3rd Edition, 2002.

c. Papoulis and S.U. Pillai, Probability Random Variables and Stochastic Processes, 4/e, McGraw-Hill, 2002.

9. URLs and Other E-Learning Resources

a. http://www.math.harvard.edu/~knill/books/KnillProbability.pdf

b. http://ocw.mit.edu/courses/mathematics/18-s096-topics-in-mathematics-with-applications-in-finance-fall-2013/assignments/MIT18_S096F13_pset2.pdf

10. Digital Learning Materials:

- http://nptel.ac.in/courses/117103067
- http://nptel.ac.in/courses/117105085
- http://192.168.0.47/Deptwise%20materials/ece/PTSP%20NOTES/
- http://192.168.0.49/videos/videoslisting/95

11. Lecture Schedule / Lesson Plan

Topic	No. of Periods	
	Theory	Tutorial
UNIT-1:REVIEW OF PROBABILITY		
Sets and set operations and events, probability space	2	1
Classical and Relative frequency approach Axiomatic Probability Theory	1	
Joint and Conditional Probabilities	1	
Total probability	1	1
Bayes theorem	2	
UNIT – 2: Random Variables		
Definition of a Random variable, Discrete and continuous	1	2
Cdf for discrete and continuous random variables	1	
Probability density functions (pdf) and properties.	2	
Binomial and Poisson distributions & Density functions	2	
Gaussian probability distribution function	1	
Uniform and Exponential density & distribution functions	2	
Rayleigh density function	1	

Expectation: mean, variance and moments of a random variable,	2	2
Characteristic functions. Properties.	1	
Moment generating functions, properties	1	
Transformations of random variable.	2	
UNIT – 3: Two Dimensional Random Variables		
Jointly distributed random variables	1	2
Conditional, joint density and distribution functions	2	
Function of two random variables; Sum of two independent RV	2	
Central limit theorem (for IID random variables),	2	2
Joint moments, covariance and correlation	2	
Independent, uncorrelated and orthogonal random variables.	1	
UNIT – 4: Classification of Random Processes		
Definition and examples – first order Random process	1	2
second order Random process	1	
strictly stationary Random process	1	
wide-sense stationary processes	1	
Ergodic processes,	2	2
Examples : white noise, Gaussian and Poisson processes.	2	
UNIT – 5: Correlation and Spectral Densities		
Auto correlation, Properties	2	2
Cross correlation, Properties	2	
Cross spectral density, Properties, Wiener-Khintchine relation	2	
Power spectral density, Properties,	2	2
Relationship between cross power spectrum and cross correlation function.	2	
UNIT – 6: Linear Systems with Random Inputs		
Linear time invariant system,	1	2
System transfer function	1	
linear systems with random inputs,	1	
Auto correlation and cross correlation functions of input and output.	2	2
Examples with white-noise as input.	1	
Total No: of Periods:	**56**	**24**

UNIT-I
REVIEW OF PROBABILITY

Objective:

- To gain the knowledge of the basic probability concepts and standard distributions which can describe real life phenomena.

Syllabus:

Sets and set operations and events, Probability space, axiomatic definition of probability, joint, conditional, total probabilities and Bayes theorem.

Learning Outcomes:

At the end of the unit student will be able to:

1) Define sample space, events, the terms related to probability theory and set theory
2) Identify the limitations of classical and relative-frequency probabilities
3) Explain probability theory in axiomatic approach
4) Determine joint and conditional probabilities
5) Apply joint and conditional probabilities for the representation of total probability
6) Distinguish dependent and independent events
7) Demonstrate Baye’s theorem

Probability theory is the branch of mathematics concerned with probability, the analysis of random phenomena.

An **experiment** is a situation involving chance or probability that leads to results called outcomes.

An **outcome** is the result of a single trial of an experiment.

An **event** is one or more outcomes of an experiment. It is a subset of sample space.

Probability is the measure of how likely an event is.

Mutually exclusive: The random experiment results in the occurrence of only one of the n outcomes. E.g. if a coin is tossed, the result is a head or a tail, but not both. That is, the outcomes are defined so as to be mutually exclusive.

Equally likely: Each outcome of the random experiment has an equal chance of occurring.

Random experiment: A random experiment is a process leading to at least two possible outcomes with uncertainty as to which will occur.

Sample space: The collection of all possible outcomes of an experiment.

Several concepts of probability have evolved over the time: They are

The classical approach

The relative frequency approach

The axiomatic approach

The classical approach

The classical approach of probability applies to equally probable events, such as the outcomes of tossing a coin or throwing dice; such events were known as "equipossible".

Probability = number of favorable equipossible / total number of relevant equipossible.

$P(A) = \frac{N(A)}{N}$, where ‘N’ is the number of all possible outcomes.

Example: If one tosses a coin there are two mutually exclusive outcomes: head or tail. Of these two outcomes, one is associated with the attribute heads; one is associated with the attribute tails. If the coin is fair each outcome is equally likely. In which case, Pr[head]= $n_A/n = 1/2$, where n=2 and n_A is the number of possible outcomes associated with a head (1).

Disadvantages:

- A Basic assumption in the definition of classical probability is that n is a finite number; that is, there are only a finite number of possible outcomes. If there are an infinite number of possible outcomes, the probability of an outcome is not defined in the classical sense.

Examples:

The roll of a die: There are 6 equally likely outcomes. The probability of each is 1/6.
Draw a card from a deck: There are 52 equally likely outcomes.
The roll of two die: There are 36 equally likely outcomes (6x6): 6 possibilities for the first die, and 6 for the second. The probability of each outcome is 1/36.

Note: An important thing to note is that classical probabilities can be deduced from knowledge of the sample space and the assumptions. Nothing has to be observed in terms of outcomes to deduce the probabilities.

The Relative Frequency Approach

What if n is not finite?
What if the outcomes are not equally likely?
In both the above cases classical definition of probability is not applicable.
In such cases, how a probability might be defined for an outcome that has event (attribute) A.

Definition: One might take a random sample from the population of interest and identify the proportion of the sample with event(attribute) A. That is, calculate

$$\text{Relative frequency of A in the sample} = \frac{\text{No. of observations in the sample that posses event}}{\text{No. of observations in the sample}}$$

Then assume "Relative freq of A in the sample" is an estimate of Pr[A].

Example: For example, one tosses a coin, which might or might not be fair, 100 times and observes heads on 52 of the tosses. One's estimate of the probability of a head is 0.52. Frequency probability allows estimating probabilities when Classical probability provides no insight.

Experimental approach, the same experiment has to be repeated n times. If the A occurs $n(A)$ times, then the relative frequency of the event A is: $\frac{n(A)}{n}$

And the probability of the event A is $P(A) = \lim_{n\to\infty} \frac{n(A)}{n}$

Limitation: Experiment has to be performed infinite number of times which may not be a feasible option.

Axiomatic Approach

The axiomatic approach builds up probability theory from a number of assumptions (axioms).

To define Axioms, there should be some sample space, i.e., the collection of all possible outcomes of an experiment.

The axiomatic approach introduces a *probability space* as its main component

Example: if the experiment is tossing a coin I = {H, T}.

Event: A subset of sample space.

For defining the value of P(A), there are nevertheless certain axioms which should always hold for internal consistency.

Axioms of probability theory

1. P(A) should be a number between 0 and 1 Or $P(A) \geq 0$
2. If A represents a certain event then P(A)=1 Or P(S)=1
3. If A_1 and A_2 are mutually exclusive events then $P(A_1 \text{ or } A_2) = P(A_1)+P(A_2)$ Or $P(\bigcup_{i=1}^{i=n} An) = \sum_{i=1}^{i=n} P(An)$

Note: if the number of outcomes is finite and equally likely then one has the Classical world of probability. Also note that the Frequency definition assumes the existence of the probability function Pr[A]. The axiomatic approach subsumes the Classical and Frequency approaches.

JOINT PROBABILITY:

A joint probability is a statistical measure where the likelihood of two events occurring together and at the same point in time is calculated. Joint probability is the probability of event Y occurring at the same time event X occurs.

P(X∩Y) = P(X) + P(Y) – P(XÚY)

Example: A joint probability cannot be calculated when tossing a coin on the same flip.

However, the joint probability can be calculated on the probability of rolling a 2 and a 5 using two different dice.

Addition Rule 1: When two events, A and B, are mutually exclusive, the probability that A or B will occur is the sum of the probability of each event.

$P(A \cup B) = P(A) + P(B)$

Example: A single 6-sided die is rolled. What is the probability of rolling a 2 or a 5?

Addition Rule 2: When two events, A and B, are not mutually exclusive, the probability that A or B will occur is $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Example: In a math class of 30 students, 17 are boys and 13 are girls. In a unit test, 4 boys and 5 girls made an A grade. If a student is chosen at random from the class, what is the probability of choosing a girl or an A student?

INDEPENDENT EVENTS:

Two events, A and B, are **independent** if the fact that A occurs does not affect the probability of B occurring.

Example: A dresser drawer contains one pair of socks with each of the following colors: blue, brown, red, white and black. Each pair is folded together in a matching set. You reach into the sock drawer and choose a pair of socks without looking. You replace this pair and then choose another pair of socks. What is the probability that you will choose the red pair of socks both times?

There are a couple of things to note about this experiment. Choosing a pairs of socks from the drawer, replacing it, and then choosing a pair again from the same drawer is a compound event. Since the first pair was replaced, choosing a red pair on the first try has no effect on the probability of choosing a red pair on the second try. Therefore, these events are independent.

Multiplication Rule 1: When two events, A and B, are independent, the probability of both occurring is:

$P(A \cap B) = P(A) \cdot P(B)$

DEPENDENT EVENTS:

Two events are **dependent** if the outcome or occurrence of the first affects the outcome or occurrence of the second so that the probability is changed.

Example: A card is chosen at random from a standard deck of 52 playing cards. Without replacing it, a second card is chosen. What is the probability that the first card chosen is a queen and the second card chosen is a jack?

The outcome of choosing the first card has affected the outcome of choosing the second card, making these events dependent.

Multiplication Rule 2: When two events, A and B, are dependent, the probability of both occurring is:

$$P(A \cap B) = P(A) \cdot P(B|A)$$

Example: A teacher needs two students to help him with a science demonstration for his class of 18 girls and 12 boys. He randomly chooses one student who comes to the front of the room. He then chooses a second student from those still seated. What is the probability that both students chosen are girls?

CONDITIONAL PROBABILITY:

Problem: A math teacher gave her class two tests. 25% of the class passed both tests and 42% of the class passed the first test. What percent of those who passed the first test also passed the second test?

Analysis: This problem describes a conditional probability since it asks us to find the probability that the second test was passed given that the first test was passed.

Let A and B are two events which are dependent, i.e., occurrence of B depends on occurrence of A, assuming A has already occurred, then the probability of both occurring is

$$P(B/A) = P(A \cap B)/P(A) \quad \text{given } P(A) \neq 0$$

Problem: The probability that it is Friday and that a student is absent is 0.03. Since there are 5 school days in a week, the probability that it is Friday is 0.2. What is the probability that a student is absent given that today is Friday?

$$P(\text{Absent}|\text{Friday}) = \frac{P(\text{Friday and Absent})}{P(\text{Friday})} = \frac{0.03}{0.2} = 0.15 = 15\%$$

Exhaustive Events:

When a sample space SS is partitioned into some mutually exclusive events such that their union is the sample space itself then the events are called exhaustive events or collectively events.

Suppose a die is tossed and the sample space is S={1,2,3,4,5,6}

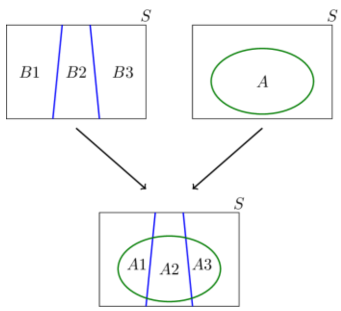
Let A={1,2}, B={3,4,5} and C={6} Hence the events A,B and C are **mutually exclusive** because A∩B∩C=ϕ and A∪B∪C=S

TOTAL PROBABILITY:

Example: In a certain country there are three provinces, call them B1, B2, and B3 (i.e., the country is partitioned into three disjoint sets B1, B2, and B3). One may be interested in the total forest area in the country. Suppose that it is known that the forest area in B1, B2, and B3 are $100km^2$, $50km^2$, and $150km^2$, respectively. What is the total forest area in the country?

Answer: $100km^2+50km^2+150km^2=300km^2$

That is, one can simply add forest areas in each province (partition) to obtain the forest area in the whole country. This is the idea behind the law of total probability, in which the *area of forest* is replaced by *probability of an event* A. In particular, if one wants to find P(A), one can look at a partition of S, and add the amount of probability of A that falls in each partition. One have already seen the special case where the partition is B and B_c: it is seen for any two events A and B,



$P(A)=P(A\cap B)+P(A\cap B_c)$

and using the definition of conditional probability,

$P(A\cap B)=P(A|B)P(B)$ we can write

$P(A)=P(A|B)P(B)+P(A|B_c)P(B_c)$

One can state a more general version of this formula which applies to a general partition of the sample space S.

Law of Total Probability:

If B1,B2,B3,⋯ is a partition of the sample space S, then for any event A we have

$P(A)=\sum P(A\cap B_i)=\sum P(A|B_i)P(B_i)$

Example:

There are three bags that each contains 100 marbles:

- Bag 1 has 75 red and 25 blue marbles;
- Bag 2 has 60 red and 40 blue marbles;
- Bag 3 has 45 red and 55 blue marbles.

One of the bags is chosen at random and then pick a marble from the chosen bag, also at random. What is the probability that the chosen marble is red?

Solution:

Let R be the event that the chosen marble is red. Let B_i be the event that the chosen Bag is i^{th} one. It is already known that

$$P(R|B_1)=0.75,\ P(R|B_2)=0.60,\ P(R|B_3)=0.45$$

One choose the partition as B_1,B_2,B_3 Note that this is a valid partition because, firstly, the B_i's are disjoint (only one of them can happen), and secondly, because their union is the entire sample space as one the bags will be chosen for sure, i.e., $P(B_1\cup B_2\cup B_3) = 1$. Using the law of total probability, one can write

$$\begin{aligned} P(R) &= P(R|B_1)P(B_1)+P(R|B_2)P(B_2)+P(R|B_3)P(B_3) \\ &= (0.75)(1/3)+(0.60)(1/3)+(0.45)(1/3) \\ &= 0.60 \end{aligned}$$

BAYES' THEOREM:

One of the most useful results in conditional probability is stated i.e., Bayes' rule. Suppose P(A|B) is known, one is interested in the probability P(B|A). Using the definition of conditional probability, one have

$P(A|B)P(B)=P(A\cap B)=P(B|A)P(A)$

Dividing by P(A), one obtains

$$P(B|A) = [\ P(A|B)P(B)\ ] / [\ P(A)\ ]$$

which is the Bayes' rule. Often, in order to find P(A) in Bayes' formula one need to use the law of total probability, so sometimes Bayes' rule is stated as

$P(B_j|A) = [\ P(A|B_j)P(B_j)\ ] / [\ \sum P(A|B_i)P(B_i)\ ]$

Where B_1, B_2 ……B_n form a partition of the sample space.

Example: In the above example, suppose it is observed that the chosen marble is red. What is the probability that Bag 1 was chosen?

Here $P(R|B_i)$ is known but P(B1|R) is desired, so this is a scenario in which one can use Bayes' rule. One have

P(B1|R)	= [P(R|B1)P(B1)] / [P(R)]
	= [0.75×0.5] / [0.6]
	=5 / 12

UNIT-II

RANDOM VARIABLES

Objective:

- To get familiarize with the one dimensional random variables and operations.
- To gain the knowledge of the standard density & distributions which can describe real life phenomena

Syllabus:

Definition of random variables, continuous and discrete random variables, cumulative distribution function (cdf) for discrete and continuous random variables; probability density functions (pdf) and properties. Expectation: mean, variance and moments of a random variable, Moment generating and characteristic functions and their properties. Binomial, Poisson, Uniform, Exponential, Gaussian and Rayleigh distributions.

Learning Outcomes:

At the end of the unit student will be able to:

1) Define the random variable
2) Distinguish continuous and discrete random variables
3) Represent cumulative and density functions
4) Calculate Nth order moments of a random variable
5) Find moments through characteristic and moment generating functions
6) Familiarize with different distribution functions

<u>Learning Material</u>

<u>Definition of random variable</u>

- A random variable X(s) is a single-valued real function that assigns a real number (called the value of X(s)) to each sample point of S.
- It is a real function of the elements of Sample Space S.
- Clearly a random variable is not a variable but it is a function.
- The sample space S is termed to be the domain of the r.v. X, and the collection of all numbers [values of X(s)] is termed as the range of the

r.v. X. Thus the range of X is a certain subset of the set of all real numbers.

- Note that two or more different sample points might give the same value of X(s), but two different numbers in the range cannot be assigned to the same sample point.

Conditions for a function to be a random variable:

1) The set $\{X \le x\}$ shall be an event for any real number x.

   This set corresponds to those points s in the sample space for which the random variable X(s) does not exceed the number x.

   The probability of this event, $P\{X \le x\}$ is equal to the sum of probabilities of all elementary events corresponding to $\{X \le x\}$.

2) Probabilities of the events $\{X = \infty\}$ and $\{X = -\infty\}$ be 0.

   $P\{X = \infty\} = 0$ and $P\{X = -\infty\} = 0$

Classification of random variabes:

1) Discrete random variable
2) Continuous random variable
3) Mixed random variable

- ***Discrete random variable*:**

A Discrete random variable is one having only discrete values. Sample space for a discrete r.v can be discrete, continuous or even a mixture of dicrete and continuous points.

- ***Continuous random variable:***

A continuous random variable is one having only continuous range of values. Sample space for a continuous r.v can't be a discrete or mixed.

X is a continuous r.v. only if its range contains an interval (either finite or infinite) of real numbers

- ***Mixed random variable:***

A mixed random variable is one having both discrete and continuous values. This is usually the least important type of random variable.

Cumulative distribution function (cdf):

The distribution function [or cumulative distribution function (cdf)] of X is a number that depends upon the real number x and hence it is a function of x.

- The function defined by

$$F_X(x) = P\{X \leq x\}$$

Where x is any real value ranging from -∞ to ∞

Properties:

1) $F_X(-\infty) = 0$
2) $F_X(\infty) = 1$
3) $0 \leq F_X(x) \leq 1$
4) $F_X(x_1) \leq F_X(x_2)$ if $x_1 < x_2$
5) $P\{x_1 < X \leq x_2\} = F_X(x_2) - F_X(x_1)$
6) $F_X(x^+) = F_X(x)$

- ***Cumulative distribution function (cdf) for discrete random variables:***

$$F_X(x) = \sum_{i=1}^{N} P(xi)u\,(x - xi)$$

Where u(x) is unit-step function; $u(x) = \begin{cases} 1 \text{ for } x \geq 0 \\ 0 \text{ for } x < 0 \end{cases}$

- ***Cumulative distribution function (cdf) for continuous random variables:***

$$F_X(x) = \int_{-\infty}^{x} fx(k)dk$$

Probability Density Functions (Pdf) and Properties:

The function $f_X(x)$ is called the probability density function (pdf) and is derivative of cdf.

$$f_X(x) = [F_X(x)]' = d/dx[F_X(x)]$$

Properties:

1) $0 \le f_X(x)$; for all x
2) $\int_{-\infty}^{\infty} fx(x)dx = 1$
3) $F_X(x) = \int_{-\infty}^{x} fx(k)dk$
4) $P\{ x_1 < X \le x_2 \} = \int_{x1}^{x2} fx(x)dx$

Distribution and Density Functions:

1) Discrete Random Variables
   i) Binomial Distribution and Density Function
   ii) Poisson Distribution and Density Function
2) Continuous Random Variables
   i) Uniform Distribution and Density Function
   ii) Exponential Distribution and Density Function
   iii) Rayleigh Distribution and Density Function
   iv) Gaussian Distribution and Density Function

DISCRETE DISTRIBUTION AND DENSITY FUNCTIONS

Binomial Random Variate:

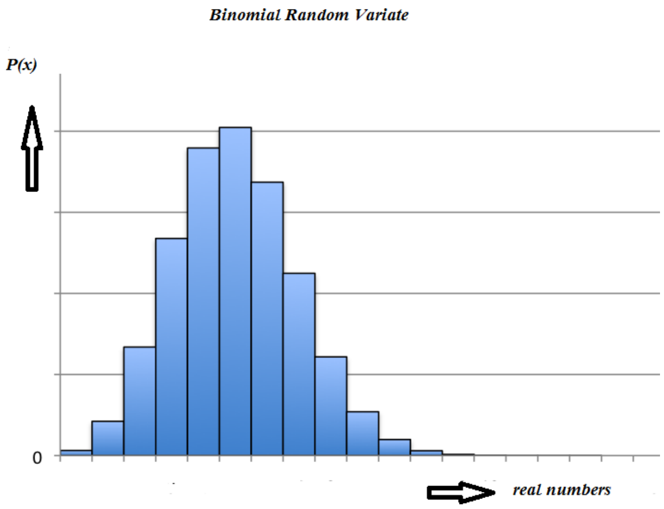
- Bernoulli Trial:

  If an event A is observed to occur repeatedly for k times out of N trials, then such repeated experiments are called Bernoulli trials
- Binomial density is applicable to Bernoulli trials.
- A random variable is said to be Binomial, if it has the following density function:

  Density Function:

  Let $0 < p < 1$ and $N = 1,2,\ldots.$, then

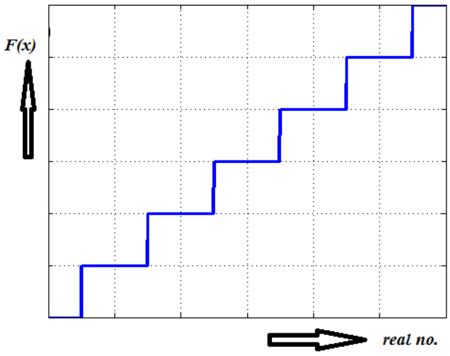
$$f_x(x) = \sum_{k=0}^{N} \binom{N}{K} p^k (1-p)^{N-k} \delta(x-k) \text{ where } \binom{N}{k} = \frac{N!}{k!(N-k)!}$$



Distribution Function

Integration of binomial density function

$$\mathbf{Fx(x)} = \sum_{k=0}^{N} \binom{N}{K} \mathbf{p^k\ (1\text{-}p)^{N\text{-}k}\ u(x\text{-}k)}$$

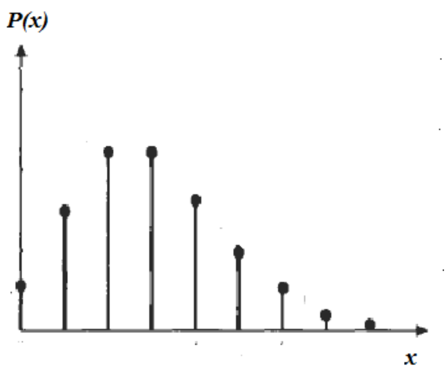


Binomial Distribution Function

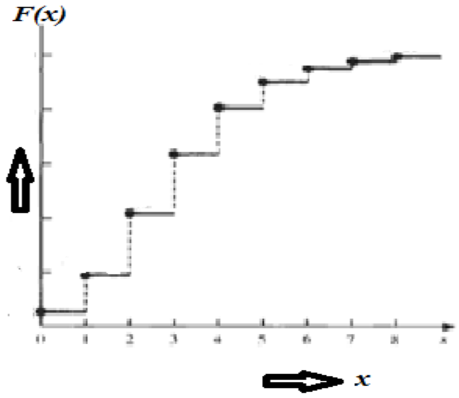
Poisson Random Variate:

Let X be a random variable and it is said to be Poisson if and only if , it has the following density function.

$$\textbf{Density Function; } f_X(x) = e^{-b} \sum_{k=0}^{\infty} \frac{b^k}{k!} \delta(x\text{-}k)$$



Distribution Function; $F_X(x) = e^{-b} \sum_{k=0}^{\infty} \frac{b^k}{k!}$ u(x-k)



Where b > 0 is a real constant;

i) b = Np if N → ∞ and p →0

ii) b = λT; λ is average rate and T is time interval of duration

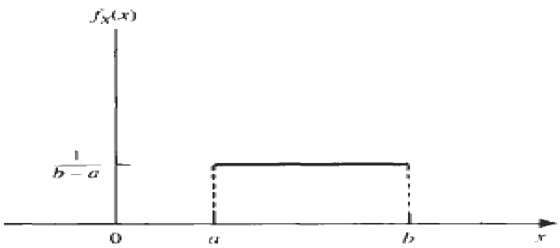
<u>CONTINUOUS DISTRIBUTION AND DENSITY FUNCTIONS</u>

<u>Uniform Random Variate:</u>

Let X be a random variable and it is said to be Uniform, if and only if , it has the following density function

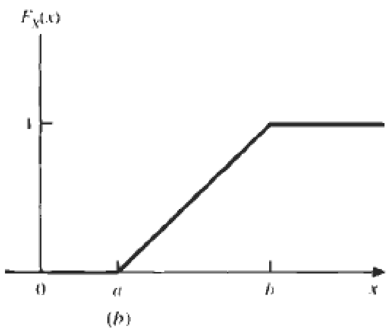
Density function

$$f_X(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{otherwise} \end{cases}$$



Distribution Function

$$F_X(x) = \begin{cases} 0 & x \le a \\ \frac{x-a}{b-a} & a < x < b \\ 1 & x \ge b \end{cases}$$

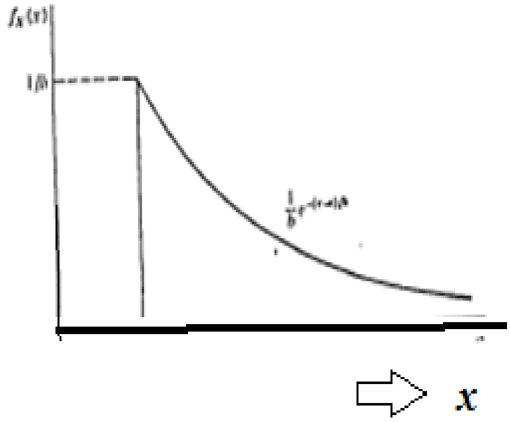


<u>Exponential Random Variate:</u>

Let X be a random variable and it is said to be Exponential, if and only if , it has the following density function

Density function:

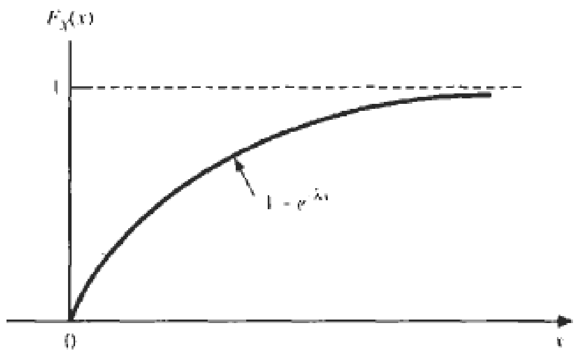
$$f_X(x) = \begin{cases} \frac{1}{b}\, e^{\frac{-(x-a)}{b}} & \textit{for } x > a \\ 0 & \textit{for } x < a \end{cases}$$



Distribution Function

$$F_X(x) = \begin{cases} 1 - e^{\frac{-(x-a)}{b}} & \text{for } x > a \\ 0 & \text{for } x < a \end{cases}$$

Fx(x) =



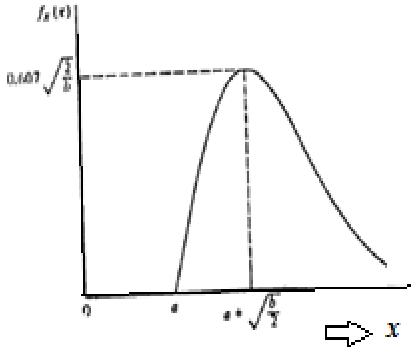
<u>Rayleigh Random Variate:</u>

Let X be a random variable and it is said to be Rayleigh, if and only if , it has the following density function

Density function:

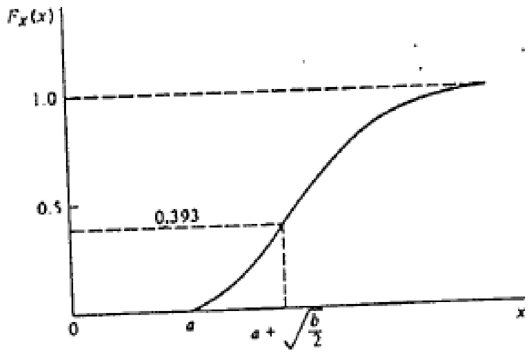
$$\mathbf{f_X(x)} \quad = \quad \frac{2}{b}\,(x-a)e^{\frac{-(x-a)2}{b}} \qquad x \geq a;$$

$$0 \qquad x < a$$



Distribution Function

$$Fx(x) = \begin{cases} \mathbf{1} - e^{\frac{-(x-a)2}{b}} & x \geq a \\ 0 & x < a \end{cases}$$



<u>Gaussian Random Variate:</u>

A random variable X is called Gaussian if its density function has the form of

$$fx\ (x) = \frac{1}{\sqrt{2\pi\sigma x2}}\ e^{\frac{-(x-ax)2}{2\sigma x2}}$$

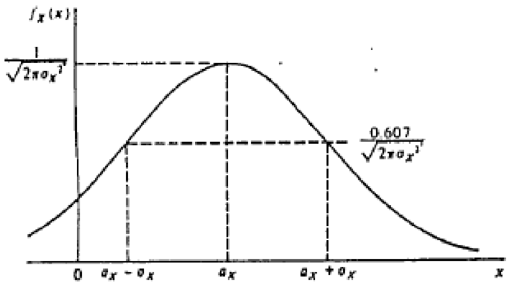
Where $\sigma x > 0$ *and* $-\infty < ax < \infty$ are real constants

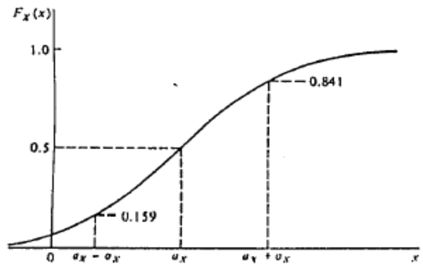
Its Maximum value $\sqrt{2\pi\sigma x2}$ occurs at x = a_x. Its spread about the point x=a_x Is related to σx. Function decreases to 0.6.7 times of its maximum at

x = a_x + σ_x and x = a_x – σ_x

Distribution function $F_X(x)$ is integral of $f_X(x)$ and is given by

$$\boldsymbol{F_X(x)} = \frac{1}{\sqrt{2\pi\sigma x2}} \int_{-\infty}^{x} e^{\frac{-(\xi - ax)2}{2\sigma x2}} \, d\xi$$





Normalized Gaussian Function

If $a_x = 0$ and $\sigma_x = 1$,then the Guassian random variate is said to be normalised.

The normalized function **F(x)** = $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{\frac{-\xi 2}{2}} \, d\xi$

This is a function of x only and x ≥ 0

For negative value of x, **F (-x) = 1 - F(x) is used.**

Let u = (ξ - ax) / σx and substitute in Fx(x); then

$$\boldsymbol{F_X(x)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{(\xi - ax)/\sigma x} e^{-u2/2} \, du$$

And is clearly equivalent to $\boldsymbol{F_X(x)} = \boldsymbol{F_X}\left(\frac{x - ax}{\sigma x}\right)$

- Function F(x) can be evaluated b approximation. F(x) = 1 – Q(x)

  Where Q(x) = $\frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{\frac{-\xi 2}{2}} \, d\xi$ is known as Q-function

- Q-approximation; Q(x) ≈ $\left[\frac{1}{(1-a)x + a\sqrt{x2 + b}}\right] \frac{e^{\frac{-x2}{2}}}{\sqrt{2\pi}}$; x ≥ 0;

  If a=0.039 and b=5.510 then **Q(x) ≈** $\left[\frac{1}{0.661x + 0.339\sqrt{x2 + 5.51}}\right] \frac{e^{\frac{-x2}{2}}}{\sqrt{2\pi}}$

Random Variable	**Mean**	**Variance**
Bernoulli	p	(1-p)
Binomial	np	np(1-p)
Poisson	λ	λ
Uniform	(a+b)/2	$(b-a)^2/12$
Rayleigh	$1/\lambda$	$1/\lambda^2$
Gaussian	μ_x	σ_x

Conditional Distribution Function:

- If an event A occurs only after the occurrence of event B, then the probability of the event P(A/B) is called Conditional Probability.

  $F_x(x|B) = P\{X \leq x|B\} = \frac{P\{X \leq x \cap B\}}{P(B)}$ where $X(s) \leq s$ and s is

subset of B

Properties:

1) $F_X(-\infty|B) = 0$
2) $F_X(\infty|B) = 1$
3) $0 \leq F_X(x|B) \leq 1$
4) $F_X(x_1|B) \leq F_X(x_2|B)$ if $x_1 < x_2$
5) $P\{x_1 < X \leq x_2|B\} = F_X(x_2|B) - F_X(x_1|B)$
6) $F_X(x^+|B) = F_X(x|B)$

Conditional Density Function:

The Conditional Density function is the derivative of Conditional Distribution function and is given by

$f_X(x|B) = [F_X(x/B)]' = d/dx\{F_X(x/B)\}$

Properties:

1) $0 \leq f_X(x|B)$; for all x
2) $\int_{-\infty}^{\infty} fx(x|B)dx = 1$
3) $F_X(x|B) = \int_{-\infty}^{x} fx(\varepsilon|B)d\varepsilon$
4) $P\{x1 < X \leq x2|B\} = \int_{x1}^{x2} fx(x|B)dx$

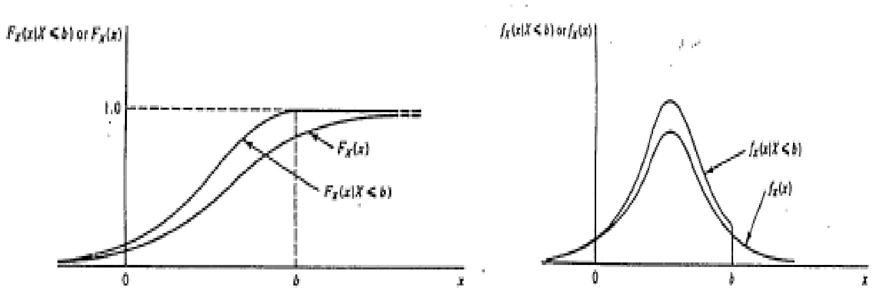
Methods of defining conditioning event:

Define event B in terms of X; so let B = {X ≤ b}; where b lies in -∞ to ∞; then

$$F_X(x|X \le B) = \begin{cases} \dfrac{F_X(x)}{F_X(b)} & x < b \\ 1 & x \ge b \end{cases}$$

$$f_X(x|X \le B) = \begin{cases} \dfrac{f_X(x)}{F_X(b)} = \dfrac{f_X(x)}{\int_{-\infty}^{b} f_X(x)dx} & x < b \\ 0 & x \ge b \end{cases}$$

Note: $F_X(x|X \le B) \ge F_X(x)$



Operations on One Random Variable:

Expected value: Defined by "Mathematical expectation of X" or "statistical average of X" or "mean value of X" or "expected value of X" and denoted by E[X] or $\bar{X}$.

- For discrete random variable $E[X] = \bar{X} = \sum_{i=1}^{N} x_i P(x_i)$
- For continuous random variable $E[X] = \bar{X} = \int_{-\infty}^{\infty} x f_X(x)dx$
- Conditional Expected value $E[X|B] = \int_{-\infty}^{\infty} x f_X(x|B)dx$

  Define event B in terms of X; so let B = {X ≤ b}; where b lies in -∞ to ∞;

  then $E[X|B] = E[X|\ X \le b] = \dfrac{\int_{-\infty}^{b} x f_X(x)dx}{\int_{-\infty}^{b} f_X(x)dx}$

Moments:

1) **Moments About Origin:**

$$\mathbf{m_n = E[X^n] = \int_{-\infty}^{\infty} x^n f_X(x)\, dx}$$

$m_0 = E[X^0] = \int_{-\infty}^{\infty} x^0 f_X(x)\, dx = 1 \rightarrow$ area of the function $f_X(x)$

$m_1 = E[X^1] = \int_{-\infty}^{\infty} x^1 f_X(x)\, dx = \bar{X} \rightarrow$expected value

2) **Central Moments:**

$$\mu_n = E[(X-\bar{X})^n] = \int_{-\infty}^{\infty} (X - \bar{X})^n f_X(x)\, dx$$

$\mu_0 = E[(X-\bar{X})^0] = \int_{-\infty}^{\infty} (x - \bar{X})^0 f_X(x)\, dx = 1 \rightarrow$ area of function $f_X(x)$

$\mu_1 = E[(X-\bar{X})^1] = \int_{-\infty}^{\infty} (x - \bar{X})^1 f_X(x)\, dx = 0$

- **Variance:** Second Central Moment

  $\mu_2 = \sigma_x^2 = E[(X-\bar{X})^2] = \int_{-\infty}^{\infty} (x - \bar{X})^2 f_X(x)\, dx = m_2 - m_1^2$

Standard Deviation (σ_x): Positive Square root of variance; it is measure of spread of $f_X(x)$ about mean.

- **Skew:** Third Central Moment

  $\mu_3 = E[(X-\bar{X})^3] = \int_{-\infty}^{\infty} (x - \bar{X})^3 f_X(x)\, dx \rightarrow$ It is a measure of asymmetry of $f_X(x)$ about $x = m_1 = \bar{X}$

Coefficient of skewness: Normalized third central moment μ_3/σ_x^3

Functions that give moments:

There are two functions that give moments. They are:

1. *Characteristic Function*
2. *Moment Generating Function*

1. **Characteristic Function:**

   i) $\boldsymbol{\Phi_x(\omega) = E[e^{j\omega x}] = \int_{-\infty}^{\infty} e^{j\omega x} f_X(x)dx;}$ $\Phi_x(\omega)$ is a Fourier Transform with $\boldsymbol{\omega}$ is reversed.

   ii) So, $f_X(x)$ can be obtained from $\Phi_x(\omega)$ by applying inverse Fourier transform; $\boldsymbol{f_x(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-j\omega x}\Phi_x(\omega)\, d\omega;}$

   iii) n^{th} moment of X is given by $\boldsymbol{m_n = (-j)^n \frac{dn}{dwn}\Phi_x(\omega)|\omega=0}$

   iv) $\Phi_x(\omega) \leq \Phi_x(0) = 1$

2. **Moment Generating Function:**

   i) $\boldsymbol{M_x(v) = E[e^{vx}] = \int_{-\infty}^{\infty} e^{vx} f_X(x)dx;}$

   ii) n^{th} moment of X is given by $\boldsymbol{m_n = \frac{dn}{dvn} M_x(v)|v=0}$

UNIT-II

Assignment Cum Tutorial Questions

Section-A:

1. If X is a random variable, 'x' being real, $x_1 < x_2 \in x$, then which of the following is true?

a) $F_X(x_1) = F_X(x_2)$ b) $F_X(x_1) \geq F_X(x_2)$ c) $F_X(x_1) \leq F_X(x_2)$ d) $F_X(x_1) \neq F_X(x_2)$

2. $\lim_{x\to\infty} F_X(x) = \lim_{x\to\infty} P\{X \leq x\}$………………. $\lim_{x\to-\infty} F_X(x)$=…………….

3.……………is monotonous increasing stair step whereas ………..is piecewise continuous function.

4. Area under the graph of pdf, $\int_{-\infty}^{\infty} f_X(x)$ is…………………………..

5. $P(a < X \leq b)$ =……………………

a) $\int_{-\infty}^{\infty} f_X(x)$ b) $\int_{b}^{a} f_X(x)$ c) $\int_{a}^{b} f_X(x)$ d) $\int_{0}^{1} f_X(x)$

6. Mean of X is the……….order moment about………..

a) I, $\bar{X}$ b) II, (0, 0) c) I, (0, 0) d) II, $\bar{X}$

7. Variance of a continuous random variable X is, Var(X) =……………

a) $E\{X-2E(X)\}^2$ b) $E\{E(X)-X\}^2$ c) $E(X^2) - \{E(X)\}^2$ d) $E(\overline{X})^2$

8. Match the following:

a) $F_X(x) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \quad n \leq x < n+1$ 1) Normal

b) $f_X(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{otherwise} \end{cases}$ 2) Binomial

c) $f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & x < 0 \end{cases}$ 3) Uniform

d) $f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/(2\sigma^2)}$ 4) Exponential

9. $f_X\{x|B\}$ =………………

10. Consider the function given by the following equation,

$$F(x) = \begin{cases} 0 & x < 0 \\ x + \frac{1}{2} & 0 \le x < \frac{1}{2} \\ 1 & x \ge \frac{1}{2} \end{cases}$$

, If X is the r.v. whose cdf is given by F(x), then $P(0<X\leq 1/4)$ is.........

a) 1/2 b)1
c)1/4 d)1/3

11. Which of the following distribution is used to approximate the number of telephone calls in a call centre switching unit at various intervals of time?

a)Binomial b)Rayleigh c)Exponential
d)Poisson

12. The second moment of a Poisson-distributed random variable is 2. The mean of the random variable is _______.

13. As x varies from – 1 to +3, which one of the following describes the behaviour of the function $f(x) = x^3 - 3x^2 + 1$?

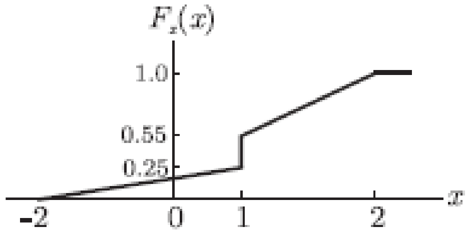
(a) f(x) increases monotonically.
(b) f(x) increases, then decreases and increases again.
(c) f(x) decreases, then increases and decreases again.
(d) f(x) increases and then decreases Then the given function f(x) increases, then decreases and increase again.

14. Let X Є {0, 1) and Y Є {0, 1} be two independent binary random variables. If P (X = 0) = *p* and P(Y = 0) = *q*, then $P(X + Y \geq 1)$ is equal to

(*a*) $pq + (1 - p)(1 - q)$ (*b*) pq
(*c*) $p(1 - q)$ (*d*) $1 - pq$

15. Let the random variable X represent the number of times a fair coin needs to be tossed till two consecutive heads appears for the first time. The expectation of X is ___________

16. The distribution function $F_X(x)$ of a random variable *x* is shown in the figure.The probability that *X* = 1 is



(a) zero (b) 0.25
(c) 0.55 (d) 0.30

17. During transmission over a certain binary communication channel, bit errors
Occur independently with probability p. The probability of *AT MOST* one bit
in error in a block of n bits is given by
(a) p^n (b) $1 - p^n$
(c) $np(- p)^n + (+ p)^n$ (d) $1 - (1 - p)^n$

18. Consider two identically distributed zero-mean random variables U and V. Let the cumulative distribution functions of U and 2V be F(x) and G(x) respectively. Then, for all values of x
(a) $F(x) - G(x) \leq 0$ (b) $F(x) - G(x) \geq 0$
(c) $(F(x) - G(x)).x \leq 0$ (d) $(F(x) - G(x)).x \geq 0$

19. Consider the experiment of tossing a coin three times. Let X be the r.v. giving the number of heads obtained. Suppose that the tosses are independent and the probability of a head is ***p*.** The range of X is........... and the probability, $P(X = 2)$ =......

a){0,1,2,3},$p(1-p)^3$ b))}{0,1,2}, $p(1-p)^3$
c))}{0,1,2,3},$3(1-p)p^2$ d))}{0,1,2},$3(1-p)p^2$

Section-B:

1. Explain the following:
   a) Continuous and discrete random variables with suitable examples.
   b) Cumulative Distribution Function and its properties, with necessary sketches.
2. a) Explain the Probability Density Function and its properties.

b)Calculate the value of the integral $I = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \exp\left(-\frac{x^2}{8}\right) dx$

3. Explain the following random variables with necessary expressions and neat sketches:

a)Binomial b) Gaussian c)Uniform

d)Exponential e)Poisson f)Rayleigh

4. a). Explain briefly the following:

i) Moments about origin ii) Moments about mean

iii) Moment Generating function and its properties

b) The moment generating function of a discrete r.v. X with $E(X^k)=0.8$, k=1,2,... is

a) $0.2+0.8e^t$ b) $0.8+0.2e^t$

c) $0.2+0.8e^{2t}$ d) $0.8+0.2e^{2t}$

5.a). Define and explain Characteristic function and its properties.

b) A random variable *X* with uniform density in the interval 0 to 1 is quantized as follows :

If $0 \le X \le 0.3$, $xq = 0$

If $0.3 < X \le 1$, $xq = 0.7$, where *xq* is the quantized value of *X*.

The root-mean square value of the quantization noise is

(a) 0.573 (b) 0.198

(c) 2.205 (d) 0.266

Problems:

1.a) A binary source generates digits 1 and 0 randomly with probabilities 0.6 and 0.4,respectively. What is the probability that at least three 1s will occur in a five-digit sequence?

i) 0.23 ii) 0.683

iii) 0.231 iv) 0.451

b). All manufactured devices and machines fail to work sooner or later. Suppose that the failure rate is constant and the time to failure (in hours) is an exponential r.v. X with parameter λ. Measurements show that the probability that the time to failure for computer memory chips in a given class exceeds 10^4 hours is e^{-t} (approx.0.368). The value of the parameter λ is.........

i) 10^4 ii) 10^{-4}

iii) 10^3 iv) 10^{-3}

2 . A) information source generates symbols at random from. a four-letter alphabet (a, *b*, ***c***, d} with probabilities P(a) = 1/2, P(b) = 1/4, and P(c) = P(d) =1/8**. A** coding scheme encodes these symbols into binary codes as follows:

a	0
b	10
c	110
d	111

Let X be the r.v. denoting the length of the code, that is, the number of binary symbols (bits).

(a) What is the range of X?

(b) Assuming that the generations of symbols are independent, find the probabilities P(X = 1),P(X = 2), P(X = **3),** and P(X > **3).**

c) Sketch the cdf $F_X(x)$ of X and specify the type of X.

Find (i) P(X ≤1), (ii) P(1 < X ≤2), (iii) P(X > 1), and (iv) P(1 ≤ X ≤ 2).

3. During transmission over a certain binary communication channel, bit errors occur independently with probability *p*. The probability of *AT MOST* one bit in error in a block of *n* bits is given by

(a) p^n (b) $1 - p^n$

(c) $np(- p)^n + (+ p)^n$ (d) $1 - (1 - p)^n$

4. A probability density function is given by p(x) = K $e^{-\frac{x^2}{2}}$ -∞<x<∞, the value of K should be

(a) $\frac{1}{\sqrt{2\pi}}$ (b) $\sqrt{\frac{2}{\pi}}$ (c) $\frac{1}{2\sqrt{\pi}}$ (d) $\frac{1}{\pi\sqrt{2}}$

5. Let X be a random variable which is uniformly chosen from the set of positive odd numbers less than 100. The expectation E[X] is __________.

$$f_X(x) = \begin{cases} \frac{1}{3} & 0 < x < 1 \\ \frac{2}{3} & 1 < x < 2 \\ 0 & \text{otherwise} \end{cases},$$

Find the corresponding cdf $F_X(x)$ and sketch $f_X(x)$ and $F_X(x)$.

UNIT III

MULTIPLE RANDOM VARIABLES

Objective:

- To familiarize with the concept of two dimensional random variables and operations.

Syllabus:

Multiple Random Variables , Independent Random Variables, Function of random variables; Central limit theorem (for IID random variables), moments, covariance and correlation;

Learning Outcomes:

At the end of the unit student will be able to:

1) Define multiple random variables
2) Represent the cumulative and density functions of multi random variables
3) Understand Central limit theorem
4) Calculate Joint moments, covariance and correlation of rvs
5) Distinguish between Correlation and Uncorrelation

Learning Material

Jointly Distributed Random Variables (Bivariate Random Variables or 2-D Random Vector):

- If X,Y are discrete r.v.s' defined on Sample Space S, then the pair (X,Y) is called Jointly ***Distributed RV or a Bivariate RV or 2D Random Vector***.
- If each of X and Y associates a real number with every element of S
- If X &Y are discrete, then {X,Y} are discrete
- If X & Y are continuous, then {X,Y} are discrete/continuous
- If one of X & Y is discrete, then {X,Y} is mixed

JOINT DISTRIBUTION FUNCTIONS

Let A= {X≤x} and B= {Y≤y} be two events defined on Sample Space and P(A) = $F_X(x)$,P(B) = $F_Y(y)$, then the *joint cumulative distribution function* (or joint cdf) of *X* and *Y* , denoted by $F_{XY}(x, y)$ is the function defined by

$$F_{XY}(x, y) = P(X \le x, Y \le y) = P(A \cap B)$$

$$F_{XY}(x, y)=\sum_{n=1}^{N} \times \sum_{m=1}^{M} P(xn, ym)u(x-xn)u(y-ym)$$

Properties of $F_{XY}(x, y)$:

The joint cdf of two r.v.'s has many properties like to those of the cdf of a single r.v.:

1. $0 \le F_{xy}(x, y) \le 1$
2. a) $F_{XY}(-\infty, -\infty) = 0$ c) $F_{XY}(-\infty, y)=0$
   b) $F_{XY}(x, -\infty)=0$
3. $F_{XY}(\infty,\infty)=1$
4. $F_{XY}(x, y)$ is a monotonic and non decreasing function of both x and y.
5. The probability of the joint event $\{x_1<X\le x_2, y_1<Y\le y_2\}$ is given by
   $P\{x_1<X\le x_2, y_1<Y\le y_2\}= F_{XY}(x_2, y_2)+ F_{XY}(x_1, y_1)- F_{XY}(x_1, y_2)- F_{XY}(x_2, y_1)$
6. The marginal distribution functions are given by $F_{XY}(x, \infty)= F_X(x)$ and $F_{XY}(\infty, y)= F_Y(y)$

Independent Random Variables:

- If X and Y are independent r.v.'s, then $p_{XY}(x_i, y_j) = p_X(x_i)p_Y(y_j)$

Continuous Random Variables-Joint Probability Density Functions:
Joint Probability Density Functions:

Let (X, Y) be a continuous bivariate r.v. with cdf $F_{XY}(x, y)$ and let

$$f_{XY}(x, y) = \frac{\partial^2 F_{XY}(x, y)}{\partial x\, \partial y}$$

In the above equation, the function $f_{XY}(x, y)$ is called the joint probability density function (joint pdf) of (X, Y) and is given by

$$F_{XY}(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{XY}(\xi, \eta)\, d\eta\, d\xi$$

Properties of $f_{XY}(x, y)$:

1. $f_{XY}(x, y) \geq 0$
2. $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x, y)\, dx\, dy = 1$
3. $f_{XY}(x, y)$ is continuous for all values of x or y except possibly a finite set.
4. $P[(X, Y) \in A] = \iint_{R_A} f_{XY}(x, y)\, dx\, dy$
5. $P(a < X \leq b, c < Y \leq d) = \int_{c}^{d} \int_{a}^{b} f_{XY}(x, y)\, dx\, dy$

Marginal Probability Density Functions:

$$F_X(x) = F_{XY}(x, \infty) = \int_{-\infty}^{x} \int_{-\infty}^{\infty} f_{XY}(\xi, \eta)\, d\eta\, d\xi$$

$$f_X(x) = \frac{dF_X(x)}{dx} = \int_{-\infty}^{\infty} f_{XY}(x, \eta)\, d\eta$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y)\, dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y)\, dx$$

- The pdf's $f_X(x)$ and $f_Y(y)$ are referred to as the marginal pdf's of X and Y, respectively.

Independent Random Variables(w.r.t cdf and pdf) :

- If X and Y are independent r.v.'s,then

$$F_{XY}(x, y) = F_X(x)F_Y(y)$$

$$\frac{\partial^2 F_{XY}(x, y)}{\partial x\, \partial y} = \frac{\partial}{\partial x} F_X(x) \frac{\partial}{\partial y} F_Y(y)$$

$$f_{XY}(x, y) = f_X(x) f_Y(y)$$

The above equation is the condition for the continuous r.v.'s X and Y are independent r.v.'s

Conditional Distributions:

Point Conditioning:

Two rvs' X and y are considered.The distribution of an rv X when the distribution function of an rv Y is known at some value of y, is defined as the conditional distribution function of X. It can be expressed as:

$$F_X(x/Y=y)=\frac{\int_{-\infty}^{x} f_{xy}(x,y)\,dx}{f_Y(y)}$$ and the conditional density function is

expressed as: $$f_X(x/Y=y) = \frac{f_{XY}(x, y)}{f_Y(y)}$$

Interval conditioning:

Assume that event b is defined in the interval $y_1<Y\leq y_2$ for the rv Y i.e., B={ $y_1<Y\leq y_2$ } and p(B) is non zero i.e., P(B)=P{ $y_1<Y\leq y_2$ }≠0, then the conditional distribution function is given by

$$F_X(x/(y_1<Y\leq y_2))=\frac{\int_{y_1}^{y_2}\int_{-\infty}^{x} f_{xy}(x,y)\,dx\,dy}{\int_{y_1}^{y_2} f_Y(y)\,dy}$$ and the conditional density function is given by

$$f_X(y/(x_1<X\leq x_2))=\frac{\int_{y_1}^{y_2} f_{xy}(x,y)\,dx}{\int_{x_1}^{x_2}\int_{-\infty}^{\infty} f_{xy}(x,y)\,dx\,dy}$$

Conditional Probability Density Functions:

If (X, Y) is a continuous bivariate r.v. with joint pdf $f_{XY}(x, y)$, then the ***conditional*** pdf of ***Y,*** given that X = x, is defined by

$$f_{Y|X}(y\mid x)=\frac{f_{XY}(x, y)}{f_X(x)} \qquad f_X(x)>0$$

Similarly,
$$f_{X|Y}(x\mid y)=\frac{f_{XY}(x, y)}{f_Y(y)} \qquad f_Y(y)>0$$

Properties of Conditional Probability Density Function:

1. $f_{Y|X}(y\mid x)\geq 0$
2. $\int_{-\infty}^{\infty} f_{Y|X}(y\mid x)\,dy=1$

As in the discrete case, if X and Y are independent, then,

$f_{Y|X}(y\mid x)=f_Y(y)$ and $f_{X|Y}(x\mid y)=f_X(x)$

EXPECTATION

Expectation of a Function of One Random Variable:

The expectation of Y = g(X) is given by

$$E(Y) = E[g(X)] = \begin{cases} \sum_i g(x_i) p_X(x_i) & \text{(discrete case)} \\ \int_{-\infty}^{\infty} g(x) f_X(x)\, dx & \text{(continuous case)} \end{cases}$$

Expectation of a Function of More than One Random Variable:

Let $X_1, \ldots, X_n$ be n r.v.'s, and let $Y = g(X_1, \ldots, X_n)$. Then

$$E(Y) = E[g(X)] = \begin{cases} \sum_{x_1} \cdots \sum_{x_n} g(x_1, \ldots, x_n) p_{X_1 \cdots X_n}(x_1, \ldots, x_n) & \text{(discrete case)} \\ \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(x_1, \ldots, x_n) f_{X_1 \cdots X_n}(x_1, \ldots, x_n)\, dx_1 \cdots dx_n & \text{(continuous case)} \end{cases}$$

COVARIANCE AND CORRELATION COEFFICIENT

The (k, n)th moment of a bivariate r.v. (X, Y) is defined by

$$m_{kn} = E(X^k Y^n) = \begin{cases} \sum_{y_j} \sum_{x_i} x_i^k y_j^n p_{XY}(x_i, y_j) & \text{(discrete case)} \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^k y^n f_{XY}(x, y)\, dx\, dy & \text{(continuous case)} \end{cases}$$

If $n = 0$, we obtain the kth moment of X, and if $k = 0$, we obtain the nth moment of Y. Thus,

$$m_{10} = E(X) = \mu_X \qquad \text{and} \qquad m_{01} = E(Y) = \mu_Y$$

$$\mu_X = E(X) = \sum_{y_j} \sum_{x_i} x_i p_{XY}(x_i, y_j)$$

$$= \sum_{x_i} x_i \left[\sum_{y_j} p_{XY}(x_i, y_j) \right] = \sum_{x_i} x_i p_X(x_i)$$

$$\mu_Y = E(Y) = \sum_{x_i} \sum_{y_j} y_j p_{XY}(x_i, y_j)$$

$$= \sum_{y_j} y_j \left[\sum_{x_i} p_{XY}(x_i, y_j) \right] = \sum_{y_j} y_j p_Y(y_j)$$

Similarly,

$$E(X^2) = \sum_{y_j} \sum_{x_i} x_i^2 p_{XY}(x_i, y_j) = \sum_{x_i} x_i^2 p_X(x_i)$$

$$E(Y^2) = \sum \sum y_j^2 p_{XY}(x_i, y_j) = \sum y_j^2 p_Y(y_j)$$

$$\mu_X = E(X) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f_{XY}(x, y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} x\left[\int_{-\infty}^{\infty} f_{XY}(x, y)\, dy\right] dx = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

$$\mu_Y = E(Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f_{XY}(x, y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} y\left[\int_{-\infty}^{\infty} f_{XY}(x, y)\, dx\right] dy = \int_{-\infty}^{\infty} y f_Y(y)\, dy$$

$$E(X^2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2 f_{XY}(x, y)\, dx\, dy = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx$$

$$E(Y^2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y^2 f_{XY}(x, y)\, dx\, dy = \int_{-\infty}^{\infty} y^2 f_Y(y)\, dy$$

The (1, 1)th joint moment of (X, Y), $m_{11} = E(XY)$ is called the correlation of X and Y. If E(XY) = 0, then

X and Y are orthogonal

The covariance of X and Y, denoted by Cov (X, Y) or σ_X,, is defined by

$$\text{Cov}(X, Y) = \sigma_{XY} = E[(X - \mu_X)(Y - \mu_Y)]$$

$$\text{Cov}(X, Y) = E(XY) - E(X)E(Y)$$

If Cov(X, Y) = **0,** then X and Y are uncorrelated

X and Y are uncorrelated if $E(XY) = E(X)E(Y)$

The correlation coefficient,denoted by ρ(X, Y) or ρ_{XY}, is defined by

$$\rho(X, Y) = \rho_{XY} = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

$$|\rho_{XY}| \le 1 \quad \text{or} \quad -1 \le \rho_{XY} \le 1$$

<u>Correlation:</u>

- If X and y are two rvs then their second order joint moment (about origin) m_{11} is called the correlation of X and Y.

$R_{XY}=m_{11}=E[XY]$

If two rvs are statistically independent, then E[XY]=E[X] E[Y]

Orthogonality:

- If two rvs X and Y are said to be orthogonal if their joint occurrence is zero.

i.e., $f_{XY}(x,y)=0$ *and* $R_{XY}=m_{11}=E[XY]=0$

FUNCTIONS OF TWO RANDOM VARIABLES

- Given two r.v.'s X and Y and a function g(x, y), the expression $Z = g(X, Y)$ defines a new r.v. Z. With ***z*** a given number,

g_{XY}(x, y) ≤ z. Then $(Z \le z) = [g(X, Y) \le z]$ and cdf is $F_Z(z) = P(Z \le z) = P[g(X, Y) \le z]$

If X and Y are continuous r.v.'s with joint pdf $f_{XY}(x, y)$, then

$$F_Z(z) = \iint_{D_z} f_{XY}(x, y)\, dx\, dy$$

Central Limit Theorem

Consider n independent random variables $X_1, X_2, ...X_n$ The mean and variance of each of the random variables are known. Suppose $E(X_i) = \mu_{X_i}$ and $\operatorname{var}(X_i) = \sigma_{X_i}^2$.

Form a random variable

$$Y_n = X_1 + X_2 + ...X_n$$

The mean and variance of Y_n are given by

$$EY_n = \mu_{Y_n} = \mu_{X_1} + \mu_{X_2} + ...\mu_{X_n}$$

and

$$\operatorname{var}(Y_n) = \sigma_{Y_n}^2 = E\{\sum_{i=1}^{n}(X_i - \mu_{X_i})\}^2$$

$$= \sum_{i=1}^{n} E(X_i - \mu_{X_i})^2 + \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} E(X_i - \mu_i)(X_j - \mu_j)$$

$$= \sigma_{X_1}^2 + \sigma_{X_2}^2 + ... + \sigma_{X_n}^2$$

$\because X_i$ **and** X_j **are independent for** $(i \neq j)$

Thus we can determine the mean and variance of Y_n.

The central limit theorem (CLT) provides an answer to this question.

The CLT states that under very general conditions $\{Y_n = \sum_{i=1}^{n} X_i\}$ converges in distribution to $Y \sim N(\mu_Y, \sigma_Y^2)$ as $n \to \infty$. The condition is:

The random variables $X_1, X_2, ..., X_n$ are independent with same mean and variance, identically distributed. The central-limit theorem can be stated as follows:

> Suppose $X_1, X_2 .... X_n$ is a sequence of independent and identically distributed random variables each with mean μ_X and variance σ_X^2 and $Y_n = \sum_{i=1}^{n} \frac{(X_i - \mu_X)}{\sqrt{n}}$. Then, the sequence $\{Y_n\}$ converges in distribution to a Gaussian random variable Y with mean 0 and variance σ_X^2. That is,
> $\lim_{n \to \infty} F_{Y_n}(y) = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}\sigma_X} e^{-u^2/2\sigma_X^2} du$

Proof of the central limit theorem

We give a less rigorous proof of the theorem with the help of the characteristic function. Further we consider each of $X_1, X_2 .... X_n$ to have zero mean. Thus,

$Y_n = (X_1 + X_2 + ... X_n)/\sqrt{n}.$

Clearly,

$\mu_{Y_n} = 0,$

$\sigma_{Y_n}^2 = \sigma_X^2,$

$E(Y_n^3) = E(X^3)/\sqrt{n}$ and so on.

The characteristic function of Y_n is given by

$$\phi_{Y_n}(\omega) = E\left(e^{j\omega Y_n}\right) = E\left[e^{\left(j\omega \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i\right)}\right]$$

We will show that as $n \to \infty$ the characteristic function ϕ_{Y_n} is of the form of the characteristic function of a Gaussian random variable.

Expanding e^{jwY_n} in power series

$e^{j\omega Y_n} = 1 + j\omega Y_n + \frac{(j\omega)^2}{2!} Y_n^2 + \frac{(j\omega)^3}{3!} Y_n^3 + ...$

Assume all the moments of Y_n to be finite. Then

$\phi_{Y_n}(\omega) = E\left(e^{j\omega Y_n}\right) = 1 + j\omega\mu_{Y_n} + \frac{(j\omega)^2}{2!} E(Y_n^2) + \frac{(j\omega)^2}{3!} E(Y_n^3) + ...$

Substituting $\mu_{Y_n} = 0$ and $E(Y_n^2) = \sigma_{Y_n}^2 = \sigma_X^2$, we get

Therefore,

$\phi_{Y_n}(\omega) = 1 - (\omega^2 / 2!)\sigma_X^2 + R(\omega, n)$

where $R(\omega, n)$ is the average of terms involving ω^3 and higher powers of ω.
Note also that each term in $R(\omega, n)$ involves a ratio of a higher moment and a power of n and therefore,

$$\lim_{n\to\infty} R(\omega, n) = 0$$

$$\therefore \lim_{n\to\infty} \phi_{Y_n}(\omega) = e^{-\frac{\omega^2\sigma_X^2}{2}}$$

which is the characteristic function of a Gaussian random variable with 0 mean and variance σ_X^2.

$$Y_n \xrightarrow{d} N(0, \sigma_X^2)$$

MOMENT GENERATING FUNCTIONS

Definition:

The *moment generating function* of a r.v. X is defined by

$$M_X(t) = E(e^{tX}) = \begin{cases} \sum_i e^{tx_i} p_X(x_i) & \text{(discrete case)} \\ \int_{-\infty}^{\infty} e^{tx} f_X(x)\, dx & \text{(continuous case)} \end{cases}$$

$$M_X(t) = E(e^{tX}) = E\left[1 + tX + \frac{1}{2!}(tX)^2 + \cdots + \frac{1}{k!}(tX)^k + \cdots\right]$$
$$= 1 + tE(X) + \frac{t^2}{2!}E(X^2) + \cdots + \frac{t^k}{k!}E(X^k) + \cdots$$

and the kth moment of X is given by

$$m_k = E(X^k) = M_X^{(k)}(0) \qquad k = 1, 2, \ldots$$

where
$$M_X^{(k)}(0) = \frac{d^k}{dt^k} M_X(t)\bigg|_{t=0}$$

Joint Moment Generating Function:

The joint moment generating function $M_{XY}(t_1, t_2)$ of two r.v.'s X and Y is defined by

$$M_{XY}(t_1, t_2) = E[e^{(t_1X + t_2Y)}]$$

The (k, n) joint moment of X and Y is given by

$$m_{kn} = E(X^k Y^n) = M_{XY}^{(kn)}(0, 0)$$

$$M_{XY}^{(kn)}(0, 0) = \frac{\partial^{k+n}}{\partial^k t_1 \partial^n t_2} M_{XY}(t_1, t_2) \Big|_{t_1 = t_2 = 0}$$

The joint moment generating function of n r.v.'s

$$M_{X_1 \cdots X_n}(t_1, \ldots, t_n) = E[e^{(t_1 X_1 + \cdots + t_n X_n)}]$$

If $X_1, X_2, X_3, \ldots . X_n$ be n independent rvs' then the moment is given by

$$\begin{aligned} M_{X_1 \cdots X_n}(t_1, \ldots, t_n) &= E[e^{(t_1 X_1 + \cdots + t_n X_n)}] = E(e^{t_1 X_1} \cdots e^{t_n X_n}) \\ &= E(e^{t_1 X_1}) \cdots E(e^{t_n X_n}) = M_{X_1}(t_1) \cdots M_{X_n}(t_n) \end{aligned}$$

Assignment-Cum-Tutorial Questions

Section-A:

Remembering / understanding level

1) Two dimensional product space is known as ---
2) The value of joint distribution function $F_{X,Y}(\infty,\infty)$ = --
3) If X and Y are statistically independent, then joint distribution function $F_{X,Y}(X,Y)$ = --
4) Central Limit theorem is mostly applicable to ---
5) If $Y = X_1+X_2+\ldots\ldots +X_N$, where $X_1,X_2, \ldots\ldots ,X_N$ are statistically independent RVs, then $f_Y(y)$ =

   a) $f_{X1}(x_1)+f_{X2}(x_2)+\ldots\ldots.+ f_{XN}(x_N)$ b) $f_{X1}(x_1)-f_{X2}(x_2)+ \ldots\ldots.+ f_{XN}(x_N)$

   c) $f_{X1}(x_1)*f_{X2}(x_2)*\ldots\ldots.*f_{XN}(x_N)$ d) $f_{X1}(x_1)\, f_{X2}(x_2)\ldots\ldots.\, f_{XN}(x_N)$
6) The distribution function of one random variable X conditioned by a second random variable Y with interval $\{ya \le y \le yb\}$ is known as

   a) moment generation b) point conditioning

   c) expectation d) interval conditioning
7) If the joint probability of X and Y is $P_{X,Y}$ (x,y) = k(x+y) for x=1,2 and y=1,2 then value of k is

   a) 1/6 b) 1/12 c) 1/3 d) ¼
8) Let X be the means of N random variables. If $Y=X_1+X_2+\ldots+X_N$, then E[Y] =

   a) $\sum_{i=1}^{N} \overline{Y_i}$ b) $\sum_{i=1}^{\infty} \overline{X_i}$ c) $\sum_{i=1}^{-\infty} \overline{X_i}$ d) $\sum_{i=1}^{N} \overline{X_i}$
9) The $(n+k)^{th}$ order joint central moment of the random variables X and Y is μ_{nk} = -----------------------------------
10) Which of the following is marginal distribution function

    a) $F_{X,Y}(\infty,Y) = F_X(X)$, b) $F_{X,Y}(\infty,Y) = F_Y(Y)$,

    c) $F_{X,Y}(\infty,Y) = f_Y(Y)$, d) $F_{X,Y}(\infty,Y) = f_X(X)$

applying / analyzing level

11) Two random variables X and Y have means 1 and 2 respectively and variance 4 and 1 respectively. Their correlation coefficient is 0.4. New random variables W and V are V = -X + 2Y, W = X + 3Y. The correlation and correlation coefficient of V and W is ……

a) 22.2 and 0.08 b) 0.222 and 0.8 c) 0.08 and 2.22 d) 2.22 and 0.8

12) Which of the relation is correct?

a) $|\sigma xy| \leq \sigma x\ \sigma y$ b) $|\sigma xy| < \sigma x\ \sigma y$ c) $|\sigma xy| \geq \sigma x\ \sigma y$ d) $|\sigma xy| > \sigma x\ \sigma y$

13) If X and Y are two independent random variables such that E[X] = λ1, variance of X is σ12, E[Y] = λ2, variance of Y is σ22, then the co-variance of [X,Y] = --------

a)σ12 σ22 + λ12 σ22 + σ12 λ22 b)σ12 σ22 + λ12 σ22 - σ12 λ22

c)σ12 σ22 - λ12 σ22 + σ12 λ22 d)σ12 σ21 + λ12 σ22 +σ21 λ22

4) For a given valid joint density function

$f_{XY}(x,y) = \begin{cases} be^{-(x+y)} & 0 < x < a; 0 < y < \infty \\ 0 & otherwise \end{cases}$, the value of constant b (in terms of a) is----

a) $\ln\frac{b}{1+b}$ b) $\ln\frac{b}{1+a}$ c) $\ln\frac{ba}{1+b}$ d) $\ln\frac{b}{1+ab}$

5) Consider an experiment of tossing a fair coin twice. Let (X,Y) be a bivariate r.v., where X is the number of heads that occurs in two tosses and Y is the number of tails that occurs in two tosses. P(X=2, Y=0), P(X=2, Y=0) and P(X=2, Y=0) are

a) ¼,¼,½ b)1,1,½ c) ½, ½, ½ d) ¼, ¼, ¼

6) The Joint cdf of a bivariate r.v (X,Y) is given by $F_{XY}(x,y)=\begin{cases}(1-e^{-\alpha x})(1-e^{-\beta y}) & x \geq 0, y \geq 0, \alpha, \beta > 0 \\ 0 & otherwise\end{cases}$, then P(X>x, Y>y) is

a) $e^{-\alpha x}+e^{-\beta y}$ b) $e^{-\alpha x}e^{-\beta y}$ c) 1,1 d) $e^{-x}e^{-y}$

7) Consider an experiment of drawing randomly three balls from an urn containing two red, three white and four blue balls. Let (X,Y) be a bivariate r.v, where X and Y denote number of red and white balls chosen. Marginal pmf's of X and Y are shown below, then A, B and C values are

Table P(X,Y)(i,j)

i	J			
	0	1	2	3
0	4/84	A	12/84	1/84
1	12/84	24/84	B	0
2	C	3/84	0	0

a) 18/84, 6/84, 4/84
b) 67/84, 72/84, 68/84
c) 18/84, 4/84,6/84
d) 4/84,6/84, 18/84

8) The Joint pdf of a bivariate r.v (X,Y) is given by

$f_{XY}(x,y)=\begin{cases}kx+y & 0<x<2, 0<y<2 \\ 0 & otherwise\end{cases}$, then P(0 < Y < ½ | X= 1) is

a) 1 b) 5/32 c) 1/8 d) 0

9) Density function of two random variables (X,Y) is $f_{XY}(x,y)= 4e^{-2(x+y)}u(x)u(y)$ then mean value of the function $e^{-(x+y)}$ is

a) 9/4 b) 3/9 c) 4/9 d) ¾

10) The correlation coefficient between X and Y from the given data is

X	1	2	3	4
P(x)	0.2	0.4	0.1	0.3

Y	1	2	3	4
P(y)	0.25	0.25	0.15	0.35

(X,Y)	(1,1)	(2,2)	(3,3)	(4,4)
P(x,y)	½	1/8	¼	1/8

a) 1.12 b) 1.2 c) -0.93 d) 0.93

Section-B:

1) Define the following:
   a) Joint probability density function b) Joint probability distribution function
2) State and prove the properties of Joint probability density function?
3) State and prove the properties of Joint probability distribution function?
4) Explain the following for two rvs'

   a) Mean b) mean square value C) skew d) skewness

   e) Correlation f) covariance

5) Determine PDF of sum of two random variables.
6) State and prove central limit theorem for i.i.d. rvs.
7) Consider an experiment of tossing a fair coin twice. Let (X, Y) be a bivariate r.v., where X is the number of heads that occurs in the first toss and Y is the number of tails that occurs in the second toss.
   (a) Find and sketch the joint density function of (X, Y).
   (b) Find and sketch the joint distribution function.

8) The joint pdf of a bivariate r.v. (X, Y) is given by

$$f_{XY}(x,y) = \begin{cases} k\,(x+y) & 0 < x < 2;\ 0 < y < 2 \\ 0 & otherwise \end{cases}$$ where k is a constant.

   (a) Find the value of k.
   (b) Find the marginal pdf's of X and Y.
   (c) Are X and Y independent?

9) Joint probability density function is $f_{X,Y}(x,y) = \begin{cases} \frac{1}{ab} \text{ for } 0 < x < a; 0 < y < b \\ 0 \quad \text{otherwise} \end{cases}$

a) Find and sketch $F_{X,Y}(x,y)$

b) if a < b, find $P\{X+Y \le 3a/4\}$

4) Statistically independent random variables X and Y have moments $m_{10}=2$, $m_{20}=14$, $m_{11}=-6$ and $m_{02}=12$. Find the moment μ_{22}.

5) Three statistically independent random variables X_1, X_2 and X_3 having mean values as 3, 6 and -2 respectively. Find the mean values of the following functions

a) $g(X_1, X_2, X_3) = X_1 + 3X_2 + 4X_3$

b) $g(X_1, X_2, X_3) = X_1 X_2 X_3$

c) $g(X_1, X_2, X_3) = -2X_1X_2 - 3X_1X_3 + 4 X_2X_3$

6) A joint density is given as $f_{X,Y}(x,y) = \begin{cases} x(y+1.5) \text{ for } 0 < x < 1; 0 < y < 1 \\ 0 \quad \text{otherwise} \end{cases}$

a) Find all the joint moments m_{nk}, n and k = 0,1,.....

b) Find all the joint central moments μ_{nk}, n and k = 0,1,.....

7) Probability density functions of two statistically independent random variables X and Y are $f_X(x) = ½\, u(x-1)e^{-(x+1)/2}$ and $f_Y(y) = ¼\, u(y-3)e^{-(y-3)/4}$. Find the probability density of the sum W = X + Y

8) Consider the binary communication channel. Let X, Y be random variables, where X is the input to the channel and Y is the output of the channel. Let P(X=0) = 0.5, P(Y=1 | X=0) = 0.1 and P(Y=0 | X=1) = 0.2

a) Find the joint pmf's of (X,Y)

b) Find the marginal pmf's of (X,Y)

c) Are X and Y are independent?

Section-C:

Previous GATE Questions:

1. Two random variables X and Y are distributed according to **[GATE-2016]**

$f_{X,Y}(x,y) = \begin{cases} (x+y), & 0 \le x \le 1, 0 \le y \le 1 \\ 0, & otherwise \end{cases}$ The probability P(X + Y ≤1) is _______

2. Let X and Y be two statistically independent random variables uniformly distributed in the ranges (-1,1) and (-2,1) respectively. Let Z = X + Y. then the probability that [Z≤-2] is

GATE-2003

(a) zero (b) $\frac{1}{6}$ (c) $\frac{1}{3}$ (d) $\frac{1}{12}$

.

UNIT IV

CLASSIFICATION OF RANDOM PROCESSES

Assignment cum tutorial questions

SECTION-A

1. Time average of x(t) is A[x(t)]= ______________
2. For an ergodic process,
   (a) Mean is necessarily zero (b) mean square value is infinity
   (c) Mean square value is independent of time (d) all time averages are zero
3. A R.P is a R.V that is a function of
   (a) Time (b) temperature (c) both (d) none
4. A stationary R.P X(t) will have its____ properties not affected by a shift in time.
   (a) Mathematical (b) normal (c) statistical (d) none
5. All strict sense stationary processes are wide sense stationary. True or false.
6. If all the statistical properties of X(t) are not affected by time shift is referred as WSS. (TRUE/FALSE)
7. _________ averages are computed by considering all the sample functions.
   (a) Time (b) ensemble (c) both (d) none.
8. If the future value of the sample function can be predicted based on its past values, then the process is referred to as
   (a) Deterministic (b) Non-deterministic (c) Independent (d) Statistical
9. For the random process X(t)= Acosωt where ω is a constant and A is uniform over (0,1) the mean square value is
   (a) 1/3 (b) (1/3)cosωt (c) $(1/3)\cos^2\omega t$ (d) 1/9
10. X(t) is a R.P defined as X(t) = cosΩt, where Ω is a uniform R.V over $(0, \omega_0)$. Then, mean of X(t) is zero at t=
    (a) $3\pi/2\omega_0$ (b) π/ω_0 (c) $\pi/2\omega_0$ (d) $\pi/4\omega_0$
11. A random process is defined as X(t) = Acos(ωt+θ), where ω and θ are constants and A is a random variable. Then, X(t) is stationary if
    (a) E[A]=2 (b) E[A]=0
    (c) A is Gaussian with non zero mean (d) A is Rayleigh with non zero mean.
12. The mean square value for the Poisson process X(t) with parameter λt is
    (a) λt (b)$(\lambda t)^2$ (c) $\lambda t + (\lambda t)^2$ (d) $\lambda t - (\lambda t)^2$

13. Difference of two independent Poisson processes is :

(a) Poisson process (b) not a Poisson process
(c) Process with mean=0 (d) Process with var = 0

14. For a Poisson random process with b=λt, the probability of exactly K occurrences over the time interval (0,t) is P[X(t)=K] is (K = 0,1,2,….)

(a) $(\lambda^t e^{-\lambda t})/K!$ (b) $(e^{-\lambda t})/K!$ (c) $(t^t e^{-\lambda t})/K!$ (d) $[(\lambda t)^t e^{-\lambda t}]/K!$

SECTION-B

1. Define a random process and classify random processes with neat sketch
2. When is a random process said to be ergodic ?
3. Define and distinguish wide sense stationary & strict sense stationary random processes.
4. Explain the following:
   a) White Noise
   b) Gaussian R.P.

Problems :

1. A random process Y(t) is given as Y(t) = X(t)cos(ωt+θ), where X(t) is a WSS R.P, 'ω' is a constant and 'θ' is a random phase independent of X(t), uniformly distributed on (π, -π). Find E[Y(t)]

2. If a random process X (t) =Acosωt + Bsinωt is given, where A and B are uncorrelated zero mean random variables having the variance σ^2. Show that X(t) is wide sense stationary.

3. A random process is given as X(t) =At, where A is an uniformly distributed random variable on (0,2). Find whether X(t) is WSS or not.

4. If $Y_1(t) = X_1\cos\omega t + X_2\sin\omega t$ and $Y_2(t) = X_1\sin\omega t + X_2\cos\omega t$ where X_1 and X_2 are zero mean independent random variables with unity variance. Show that the random processes $Y_1(t)$ and $Y_2(t)$ are individually WSS.

5. A random process Y(t) = X(t)-X(t+τ) is defined in terms of a process. X(t) that is at least WSS. (a) show that mean value of Y(t) s zero even if X(t) has a non zero mean value. (b) If Y(t) =X(t)+X(t+τ). Find E(Y(t)] and σ^2 of Y.

6. A random process is defined by X(t)=A, where A is a continuous random variable uniformly distributed on (0,1). (a) Classify the process. (b) Is it deterministic?

SECTION-C

1. X(t) is a random process with a constant mean value of 2 and the autocorrelation function $R_x(\tau) = 4\left[e^{-0.2|\tau|} + 1\right]$. **GATE - 2003**

a) Let X be the Gaussian random variable obtained by sampling the process at t = t_i and let $Q(\alpha) = \int_{\alpha}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-y^2}{2}} dy$.

The probability that $[x \le 1]$ is

(a) 1 – Q(0.5) (b) Q(0.5) (c) $Q\left(\frac{1}{2\sqrt{2}}\right)$ (d) $1 - Q\left(\frac{1}{2\sqrt{2}}\right)$

b) Let Y and Z be the random variables obtained by sampling X(t) at t =2 and t = 4 respectively. Let W = Y – Z. The variance of W is

(a) 13.36 (b) 9.36 (c) 2.64 (d) 8.00

UNIT V

RANDOM PROCESS(SPECTRAL CHARACTERISTICS)

Objective:

- To understand and apply the concept of random processes in the analysis and processing of the signals and noise

Syllabus:

Power Spectral Density & its Properties, Cross Spectral Density & its Properties, Wiener Khintchine Relation, (Relationship between Power Spectrum and Auto Correlation Function)

Learning Outcomes:

At the end of the unit student will be able to:

1) Define power density spectrum and its properties
2) Explain the cross spectral density and its properties
3) Explain Wiener Khintchine Relation

<u>Learning Material</u>

Power Spectral Density:

The *power spectral density* (or *power spectrum) Sx(w)* of a continuous-time random process *X(t)* is defined as the Fourier transform of $R_X(\tau)$:

$$S_X(\omega) = \int_{-\infty}^{\infty} R_X(\tau) e^{-j\omega\tau}\, d\tau$$

, Thus, taking the inverse Fourier transform of *Sx(w),one gets*

$$R_X(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega) e^{j\omega\tau}\, d\omega$$

The above two relations are known as the ***Wiener-Khinchine relations***.

Properties of $S_X(\omega)$:

1. $S_X(\omega)$ is real and $S_X(\omega) \geq 0$.
2. $S_X(-\omega) = S_X(\omega)$
3. $E[X^2(t)] = R_X(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega)\, d\omega$

—

Similarly, the power spectral density Sx(Ω) of a discrete-time random process X(n) is defined as the
Fourier transform of Rx(k):

$$S_X(\Omega) = \sum_{k=-\infty}^{\infty} R_X(k) e^{-j\Omega k}$$

Thus, taking the inverse Fourier transform of Sx(Ω)

$$R_X(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_X(\Omega) e^{j\Omega k}\, d\Omega$$

Properties of $S_X(\Omega)$:

1. $S_X(\Omega + 2\pi) = S_X(\Omega)$
2. $S_X(\Omega)$ is real and $S_X(\Omega) \geq 0$.
3. $S_X(-\Omega) = S_X(\Omega)$
4. $E[X^2(n)] = R_X(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_X(\Omega)\, d\Omega$

Cross Power Spectral Densities:
The cross power spectral density (or cross power spectrum) Sxy(w) of two continuous-time random
processes X(t) and Y(t) is defined as the Fourier transform of $R_{XY}(\tau)$:

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau) e^{-j\omega\tau}\, d\tau$$

Thus, taking the inverse Fourier transform of Sx(w),one gets,

$$R_{XY}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XY}(\omega) e^{j\omega\tau}\, d\omega$$

Properties of $S_{XY}(w)$:

Unlike Sx(w), which is a real-valued function of w, Sxy(w), in general, is a complex-valued function.

1. $S_{XY}(\omega) = S_{YX}(-\omega)$
2. $S_{XY}(-\omega) = S^*_{XY}(\omega)$

Similarly, the cross power spectral density S_{XY}(w) of two discrete-time random processes X(n) and
Y(n) is defined as the Fourier transform of Rxy(τ):

$$S_{XY}(\Omega) = \sum_{k=-\infty}^{\infty} R_{XY}(k)e^{-j\Omega k}$$

Thus, taking the inverse Fourier transform of Sxy(Ω),

$$R_{XY}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_{XY}(\Omega)e^{j\Omega k}\, d\Omega$$

Properties of S_{XY} (Ω):

Unlike ***S_X (Ω),*** which is a real-valued function of ***w,*** Sxy(Q), in general, is a complex-valued function

1. $S_{XY}(\Omega + 2\pi) = S_{XY}(\Omega)$
2. $S_{XY}(\Omega) = S_{YX}(-\Omega)$
3. $S_{XY}(-\Omega) = S^*_{XY}(\Omega)$

Weiner khintzine Relation:

This states the relationship between the Power spectral density and Auto Correlation function.

The inverse Fourier Transform of Power Spectral density is equal to Time average of Auto Correlation Function.The proof of it is given below:

Relationship between Power Spectrum and autocorrelation function

The inverse Fourier Transform of Power Spectral Density is equal to Time Average of Auto Correlation Function and it is expressed as:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} Sxx(\omega)e^{j\omega t}d\omega = A[Rxx(t,t+T)]$$

Consider L.H.S.,and in that consider the power spectral density, the formula for power spectral density is,

$$Sxx(\omega)=\lim_{T\to\infty} E[\frac{1}{2\pi}\int_{-T}^{T} X(t\,1)e^{j\omega t1}dt1 \int_{-T}^{T} X(t2)e^{-j\omega t2}dt2]$$

$$=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T}E[X(t1)X(t2)]e^{-j\omega(t2-t1)}dt2\,dt1$$

E[x(t1)X(t2)]=Rxx(t1,t2) -T<(t1 and t2)<T

$$Sxx(\omega)=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T}Rxx(\mathrm{t1,t2})\ \mathrm{e}^{-\mathrm{j}\omega(\mathrm{t2-t1})}\mathrm{dt1\ dt2}$$

Substitute the above equation in actual L.H.S.,

$$=>\quad \frac{1}{2\pi}\int_{-T}^{T}Sxx(\omega)\mathrm{e}^{\mathrm{j}\omega\mathrm{t}}\,d\omega$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T}\mathrm{R}xx(t,t1)e^{-j\omega(t-t1)}dtdt1e^{j\omega t}d\omega$$

$$=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T}\mathrm{R}xx(t,t1)\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{j\omega(\tau-t1+t)}d\omega\,dt1\,dt$$

$$=>\ \frac{1}{2\pi}\int_{-\infty}^{\infty}Sxx(\omega)\mathrm{e}^{\mathrm{j}\omega\tau}\,d\omega\qquad=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T}\mathrm{R}xx(\mathrm{t,t1})\delta(\mathrm{t1-t-\tau})\,dt1\,dt$$

Integration of Impulse within the interval assumes the value of unity.

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}Sxx(\omega)\mathrm{e}^{\mathrm{j}\omega\tau}\,\mathrm{d}\omega$$

$$=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}Rxx(\mathrm{t,t+\tau})\mathrm{dt}\qquad -\mathrm{T} < \mathrm{t}+\tau < \mathrm{T}$$

$$A[Rxx(t,t+\tau)] = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\mathrm{R}xx(t,t+\tau)\ \mathrm{dt}$$

$$Sxx(\omega)=\int_{-\infty}^{\infty}A[\mathrm{R}xx(t,t+\tau)]e^{-j\omega\tau}d\tau$$

A[Rxx(t,t+τ)] <--> Sxx(ω)(Hence Proved)

$$Sxx(\omega) = \int_{-\infty}^{\infty}\mathrm{R}xx(\tau)\ e^{-j\omega\tau}\ d\tau$$

$$Rxx(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}Sxx(\omega)\mathrm{e}^{\mathrm{j}\omega\tau}\,\mathrm{d}\omega$$

Or They form Fourier Transform pairs.

$$Rxx(\tau) <--> Sxx(\omega)$$

Assignment-Cum-Tutorial Questions

Questions testing the remembering / understanding level of students

<u>Section-A</u>

Objective Questions:

1. The power density spectrum of power spectral density of $X_T(t)$ =X(t) for –T<t<T; $X_T(t)$ =0 , elsewhere is $S_{XX}(\omega)$ =

   a) $\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{|X_T(\omega)|^2}{2T}$ b) $\int_{-\infty}^{\infty}\frac{|X_T(\omega)|^2}{2T}d\omega$ c) $\lim_{T\to\infty}\frac{E\{|X_T(\omega)|^2\}}{2T}$ d) $\frac{1}{2\pi}\lim_{T\to\infty}\frac{E\{|X_T(\omega)|^2\}}{2T}$

2. The average power of the random process having PSD Sxx(ω) is Pxx=

   a) zero b) $\int_{-\infty}^{\infty}S_{XX}(\omega)d\omega$ c) $\frac{1}{2\pi}\int_{-\infty}^{\infty}S_{XX}(\omega)d\omega$ d) $2\pi\int_{-\infty}^{\infty}S_{XX}(\omega)d\omega$

3. The time average of the autocorrelation function and the power spectral density form a pair of

   a) Fourier transform b) Laplace transform c) Z-transform d) convolution

4. The power spectral density of WSS is always

   a) negative b) non-negative c) finite d) can be negative or positive

5. For a WSS process, PSD at zero frequency gives

   a) the area under the graph of a power spectral density

   b) area under the graph of the autocorrelation of the process

   c) mean of the process

   d) variance of the process

6. The cross spectral density Syx(ω) =

   a) Sxy(ω) b) Sxy(-ω) c) Syx(-ω) d) –Syx(ω)

7. The cross power Pxy is given by

   a) $\int_{-\infty}^{\infty}S_{XY}(\omega)d\omega$ b) $2\pi\int_{-\infty}^{\infty}S_{XY}(\omega)d\omega$ c) $\frac{1}{2\pi}\int_{-\infty}^{\infty}S_{YX}(\omega)d\omega$ d) $\frac{1}{2\pi}\int_{-\infty}^{\infty}S_{XY}(\omega)d\omega$

8. A random process has the PSD Sxx(ω) = $\frac{6\omega^2}{1+\omega^2}$.

Find the average power in the process.

a) 1.06 W b) 2.06 W c) 0.06W d) 0W

9. If Sxx(ω) = $\frac{16}{16+\omega^2}$ Syy(ω) = $\frac{\omega^2}{16+\omega^2}$ and X(t) and Y(t) are of zero mean, U(t) = X(t) + Y(t),

Sxu(ω) is

a) $\frac{\omega^2}{16+\omega^2}$ b) $\frac{\omega}{16+\omega^2}$ c) $\frac{4\omega}{16+\omega^2}$ d) $\frac{16}{16+\omega^2}$

10. A WSS process, X(t), has an autocorrelation function Rxx(τ)=$e^{-3|\tau|}$. Then PSD is

a) $\frac{6}{9+\omega^2}$ b) $\frac{9}{6+\omega^2}$ c) $\frac{3}{9+\omega^2}$ d) $\frac{9}{3+\omega^2}$

Section B

Question testing the ability of students in analyzing/applying the concepts

1. Define power spectral density and explain its properties.

2. Define cross power spectral density and explain its properties

3. Is the sum process a discrete-time stationary process?

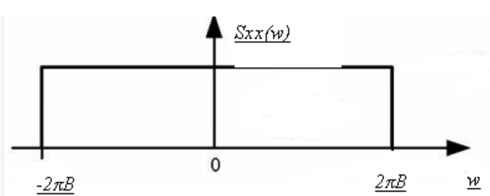
4. If Sxx(ω) = η/2 for -2πB≤ω≤2πB, then Rxx(τ) is ________

a) ηBsinc(2πBτ) b) Bsinc(2πBτ) c) sinc(2πBτ)/ηB d) sinc(2πBτ)/B4

5. Consider a WSS random process X(t) with $R_X(\tau)=e^{-a|\tau|}$, where a is a positive real number. Find the PSD of X(t).

6. Determine the auto correlation function Rxx(τ) and the power of the low pass random process

with power spectral density Sxx(ω) = η/2, which is shown as



7. The cross spectral density of two random process X(t) and Y(t) is

Sxy(ω) = $1+\frac{J\omega}{K}$ for –k<w<k, and zero elsewhere, where k>0. Find the cross correlation function between the processes.

8. Find the autocorrelation function of:

a) $S_{XX}(\omega)=\frac{157+12\omega^2}{(16+\omega^2)(9+\omega^2)}$ b) $S_{XX}(\omega)=\frac{8}{(9+\omega^2)^2}$

9. The autocorrelation function of a WSS process is R(τ) = $Ke^{-k|\tau|}$. Then which of the following is its spectral density ,S(ω) ?

a) $\frac{1}{1+\left(\frac{\omega}{K}\right)^2}$ b) $\frac{2}{1+\left(\frac{\omega}{K}\right)^2}$ c) $1+\left(\frac{\omega}{K}\right)^2$ d) $[1+\left(\frac{\omega}{K}\right)^2]/2$

UNIT VI

LINEAR SYSTEMS WITH RANDOM INPUTS

Objective:

To understand and describe the response of a linear system when the random processes and white-noise are applied.

Syllabus:

Linear time invariant system, Stationary of output, Auto correlation and power spectral density of output. Examples with white-noise as input

Learning Outcomes:

At the end of the unit student will be able to:

1) Define the response of a Linear Time Invariant(LTI) system and its System Transfer Function
2) Express Mean and Mean-Squared Value of System Response
3) Express Auto correlation and Cross correlation functions of input and output
4) Express Power Density Spectrum and Cross Power Density Spectrum of System Response
5) Calculate System Response of the functions(Mean, Mean-Squared Value, Auto correlation, Cross correlation, Power Density Spectrum and Cross Power Density Spectrum) when white –noise is an input

LEARNING MATERIAL

6.1 Linear System:

In general, the linear system will cause the response y(t) operating on x(t) and is represented as **y(t) = L[x(t)]**; where L is an operator representing the action of system on x(t).

A system is said to be linear if its response to a sum of inputs $x_n(t)$, n=1,2,….,N is equal to the sum of responses taken individually.

Thus if $x_n(t)$ causes a response $y_n(t)$, n=1,2,….,N, then for a linear system;

$$y_n(t) = L[x_n(t)]$$

$$= L[\sum_{n=1}^{N} \alpha_n x_n(t)]$$

$$= \sum_{n=1}^{N} L[\alpha_n x_n(t)]$$

$$\mathbf{y_n(t) = \sum_{n=1}^{N} \alpha_n y_n(t)}$$ must valid;

where α_n are arbitrary constants and N may be infinite.

From the definition and by the property of impulse function; x(t) is represented as $\mathbf{x(t)} = \int_{-\infty}^{\infty} \mathbf{x(\xi)\delta(t-\xi)d\xi}$.

By substituting x(t) in y(t) = L[x(t)];

$$y(t) = L[\int_{-\infty}^{\infty} \mathbf{x(\xi)\delta(t-\xi)d\xi}]$$

and let the operator operates on the time function; then

$$y(t) = \int_{-\infty}^{\infty} \mathbf{x(\xi)\ L\delta(t-\xi)d\xi};$$

let $\mathbf{L\delta(t-\xi) = h(t,\xi)}$ and is defined as the impulse response of linear system.

$$\mathbf{y(t)} = \int_{-\infty}^{\infty} \mathbf{x(\xi)\ h(t,\xi)d\xi}.$$

The above expression is the output response of a linear system and is completely determined by its impulse response.

6.2 Linear Time-Invariant System:

A general linear system is said to be linear time-invariant if the impulse response $h(t,\xi)$ does not depend on the time that the impulse is applied.

Thus if an impulse $\boldsymbol{\delta}$(t), occurring at t=0, causes the response h(t), then an impulse $\boldsymbol{\delta}$(t- ξ), occurring at t= ξ must causes the response h(t- ξ) if the system is time-invariant; i.e. $\mathbf{h(t,\xi) = h(t-\xi)}$.

Hence output response of a linear time-invariant system is given by

$$\mathbf{y(t)} = \int_{-\infty}^{\infty} \mathbf{x(\xi)\ h(t-\xi)d\xi}$$

and is also known as convolution integral of $\mathbf{x(t)}$ and $\mathbf{h(t)}$ and is given by

$$\mathbf{y(t)} = \mathbf{x(t) * h(t)} = \int_{-\infty}^{\infty} \mathbf{x(\xi)\ h(t-\xi)d\xi}$$

by a suitable change of variables, alternative form defined below is also valid

$$\mathbf{y(t)} = \mathbf{x(t) * h(t)} = \int_{-\infty}^{\infty} \mathbf{x(t-\xi)\ h(\xi)d\xi}$$

6.3 Time-Invariant System Transfer Function [H(ω)]:

H(ω), Y(ω) and X(ω) are the respective Fourier transforms of h(t), y(t) and x(t).

$$\mathbf{Y(\omega)} = \mathbf{F[y(t)]} = \int_{-\infty}^{\infty} \mathbf{y(t)\ e^{-jwt}\,dt}$$

$$= \int_{-\infty}^{\infty}[\int_{-\infty}^{\infty} x(\xi)\ h(t-\xi)d\xi]\ e^{-jwt}\,dt$$

$$= \int_{-\infty}^{\infty}[\int_{-\infty}^{\infty} x(\xi)\ h(t-\xi)d\xi]\ e^{-jw(\xi)}\ e^{-jw(t-\xi)}dt$$

$$= \int_{-\infty}^{\infty} x(\xi)\ e^{-jw(\xi)}d\xi \int_{-\infty}^{\infty} h(t-\xi)\ e^{-jw(t-\xi)}dt$$

$$\mathbf{Y(\omega)} = \mathbf{X(\omega)\ H(\omega)}$$

Where the function **H(ω)** is called the transfer function of the system. Hence Fourier transform for the response of any linear time-invariant system is the product of transfer function of input signal and transfer function for the network impulse response.

Causal System:

A linear time-invariant system is said to be causal if it does not respond to the application of input signal. Mathematically this implies that y(t) = 0 for $t < t_0$ if x(t) = 0 for $t < t_0$ and requires that h(t) = 0 for t < 0; where t_0 is any real constant

Stable System:

A linear time-invariant system is said to be stable if its response to any bounded input is bounded; i.e. if |x(t)| < M then |y(t)| < MI for a stable system; where M is a constant and I is a constant independent of input and given by $I = \int_{-\infty}^{\infty} |h(t)| dt < \infty$; having the impulse response h(t) will be stable.

6.4 Linear Systems With Random Inputs:

6.4.1 System Response-Convolution:

- When x(t) is a random signal and h(t) is the networks impulse response then the networks response y(t) is given the convolution integral. $\mathbf{y(t) = x(t) * h(t)}$.
- If a random process X(t) is applied to a linear time-invariant system whose impulse response is h(t) then it produces a new random process Y(t); given by $\mathbf{Y(t) = X(t) * h(t) = \int_{-\infty}^{\infty} X(t-\xi)\, h(\xi) d\xi}$

6.4.2 Mean Value of System Response:

Consider X(t) is a WSS; then Mean Value of System Response is obtained by;

$$E[Y(t)] = E[\int_{-\infty}^{\infty} X(t-\xi)\, h(\xi) d\xi]$$

$$= \int_{-\infty}^{\infty} h(\xi)\, E[X(t-\xi)]\, d\xi]$$

$$= \bar{X} \int_{-\infty}^{\infty} h(\xi) d\xi$$

$$\mathbf{\bar{Y} = \bar{X} \int_{-\infty}^{\infty} h(\xi) d\xi}$$

Thus mean value of system response Y(t) equals the mean value of X(t) times the area under the impulse response if X(t) is a WSS.

6.4.3 Mean-Squared Value of System Response:

Mean-Squared Value of System Response is obtained by;

$$E[Y^2(t)] = E[X(t) * h(t)^2] = E[[X(t) * h(t)][X(t) * h(t)]]$$

$$= E[\int_{-\infty}^{\infty} h(\xi 1) X(t-\xi 1) d\xi 1 \int_{-\infty}^{\infty} h(\xi 2) X(t-\xi 2)]\, d\xi 2]$$

$$= E[\iint_{-\infty}^{\infty} X(t-\xi 1)\, X(t-\xi 2) h(\xi 1) h(\xi 2) d\xi 1 d\xi 2]$$

$$= \iint_{-\infty}^{\infty} E[X(t-\xi 1)\, X(t-\xi 2)] h(\xi 1) h(\xi 2) d\xi 1 d\xi 2$$

Consider input is WSS; then $\mathbf{E[X(t-\xi 1)X(t-\xi 2)] = R_{xx}(\xi_1\text{-}\xi_2)}$; hence

$$\mathbf{E[Y^2(t)] = \iint_{-\infty}^{\infty} Rxx(\xi 1-\xi 2) h(\xi 1) h(\xi 2) d\xi 1 d\xi 2}$$

6.4.4 Auto Correlation Function Of Input And Output:

Let X(t) be WSS; then autocorrelation function of Y(t) is obtained by

$$R_{YY}(t, t+\tau) = E[Y(t)\, Y(t+\tau)]$$

$$= E[\int_{-\infty}^{\infty} h(\xi 1) X(t-\xi 1) d\xi 1 \int_{-\infty}^{\infty} h(\xi 2) X(t+\tau-\xi 2)]\, d\xi 2]$$

$$= \iint_{-\infty}^{\infty} E[X(t-\xi 1)\, X(t+\tau-\xi 2)] h(\xi 1) h(\xi 2) d\xi 1 d\xi 2$$

As X(t) is assumed WSS, then the above expression reduces to

$$R_{YY}(\tau) = \iint_{-\infty}^{\infty} R_{XX}(\tau+\xi 1-\xi 2) h(\xi 1) h(\xi 2) d\xi 1 d\xi 2$$

1) Y(t) is WSS if X(t) is WSS because $R_{YY}(\tau)$ does not depend on t and E[Y(t)] is a constant.
2) $R_{YY}(\tau)$ is the two-fold convolution of the input autocorrelation function with the networks impulse response given by

$$\mathbf{R_{YY}(\tau) = R_{XX}(\tau) * h(-\tau) * h(\tau)}$$

6.4.5 Cross Correlation Function Of Input And Output:

The cross-correlation function of X(t) and Y(t) is

$$R_{XY}(t, t+\tau) = E[X(t)\, Y(t+\tau)]$$

$$= E[X(t) \int_{-\infty}^{\infty} h(\xi) X(t+\tau-\xi) d\xi]$$

$$= \int_{-\infty}^{\infty} E[X(t) X(t+\tau-\xi)] h(\xi) d\xi$$

If X(t) is WSS, the above expression reduces to

$$R_{XY}(\tau) = \int_{-\infty}^{\infty} R_{XX}(\tau-\xi) h(\xi) d\xi$$

Which is the convolution $R_{XX}(\tau)$ with $h(\tau)$

$$\mathbf{R_{XY}(\tau) = R_{XX}(\tau) * h(\tau)}$$

Similarly we will obtain

$$\mathbf{R_{YX}(\tau) = \int_{-\infty}^{\infty} R_{XX}(\tau-\xi) h(-\xi) d\xi = R_{XX}(\tau) * h(-\tau)}$$

Relation between autocorrelation and cross correlation:

From the expressions of autocorrelation function it is clear that

$$\mathbf{R_{YY}(\tau) = R_{XY}(\tau) * h(-\tau)}$$

$$R_{YY}(\tau) = R_{YX}(\tau) * h(\tau)$$

6.4.6 Power Density Spectrum of Response:

Power density spectrum $\varphi_{YY}(\omega)$ of the response of a linear time-invariant system having a transfer function $H(\omega)$ is given by

$$\varphi_{YY}(\omega) = \varphi_{XX}(\omega)|H(\omega)|^2$$

where $\varphi_{XX}(\omega)$ is the power spectrum of input process $X(t)$ and $|H(\omega)|^2$ is the power transfer function of the system.

6.4.7 Cross-Power Density Spectrum of Input and Output Response:

Cross-Power density spectrum $\varphi_{XY}(\omega)$ and $\varphi_{YX}(\omega)$ of the response of a linear time-invariant system having a transfer function $H(\omega)$ are given by;

$$\varphi_{XY}(\omega) = \varphi_{XX}(\omega)\, H(\omega)$$

$$\varphi_{YX}(\omega) = \varphi_{XX}(\omega)H(-\omega) \text{ respectively}$$

6.5 Examples with white-noise as input

By substituting the input power spectral density with $\varphi_{XX}(\omega) = \eta/2$,it is possible to analyze the system with white noise as an input.

Assignment-Cum-Tutorial Questions

Section-A:

Remembering / understanding level

1) A system is said to be linear system if it satisfies

   a) Principle of superposition c) Principle of homogeneity

   b) a and c d) Reciprocity principle

2) For an LTI system, the response y(t) for any input x(t), with impulse response is determined by using the following integral

   a) Convolution b) Fourier c) Laplace d) b and c

3) The cross correlation between X(t) and y(t) is $R_{XY}(\tau)$ =

   a) $h(\tau)* R_{XX}(\tau)$ b) $h(-\tau)* R_{XX}(\tau)$ c) $h(-\tau)* R_{XY}(\tau)$ d) $h(\tau)* R_{YX}(\tau)$

4) A random process X(t) of mean 3 is applied to a delay element. The mean of output process is

   a) 2 b) 3 c) 1.5 d) 9

5) If $P_{XX}(\omega)$ is the power spectrum of the input response X(t) and $|H(\omega)^2|$ is the transfer function of the system, then the average power P_{YY} is =

6) The output power density of Y(t) can be obtained by, $P_{YY}(\omega)$ =

7) The cross power density of X(t) and Y(t) can be obtained by $P_{YX}(\omega)$ =

   a) $H(\omega)P_{XX}(\omega)$ b) $H^*(\omega)P_{XX}(\omega)$ c) $H(\omega)/P_{XX}(\omega)$ d) $H^*(\omega)/P_{XX}(\omega)$

applying / analyzing level

8) A stationary random process X(t) has auto-correlation function $R_{XX}(\tau) = 10 + 5\cos(2\tau) + 10e^{-2|\tau|}$; then dc average power of X(t) is

   a) 10W b) 15W c) 25W d) 20W

9) A stationary random process X(t) has auto-correlation function $R_{XX}(\tau) = 10 + 5\cos(2\tau) + 10e^{-2|\tau|}$; then average power of X(t) is

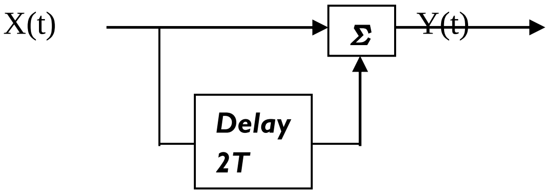
   b) 10W b) 15W c) 25W d) 20W

10) Mean Square Value of the output response for a system having $h(t)=e^{-t}u(t)$ and input of white noise with psd $N_o/2$ is

   a) $N_o/2$ b) $N_o/4$ c) $N_o/8$ d) $N_o/16$

11) The psd of a random process is given by $\varphi_{XX}(\omega)=\frac{16}{16+w2}$. Is it a valid psd. State True or False.

12) A WSS RP X(t) with psd $\Phi_{XX}(\omega)$ is applied at the input of a delay system as shown below; then psd of Y(t) is



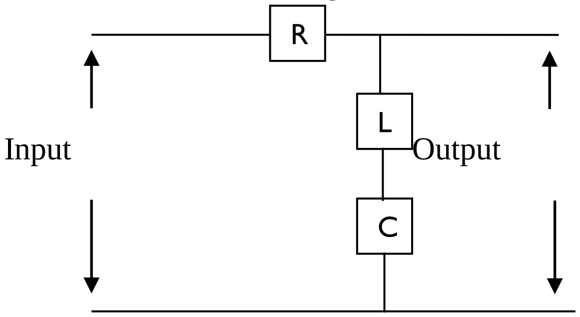
a) $2\cos^2(\omega T)\ \Phi_{XX}(\omega)$

b) $2\cos^2(\omega T)\ \Phi_{XX}(\omega)$

c) $4\cos^2(\omega T)\ \Phi_{XX}(\omega)$

d) $4\cos^2(\omega T)\ \Phi_{XX}(-\omega)$

13) Transfer function for the following circuit is



a) $(1+\omega^2RC)/(1-\omega^2LC+j\omega RC)$

b) $(1-\omega^2RC)/(1-\omega^2LC+j\omega RC)$

c) $(1-\omega^2RC)/(1+\omega^2LC+j\omega RC)$

d) $(1+\omega^2RC)/(1+\omega^2LC+j\omega RC)$

14) A stationary random process X(t) and Y(t) has a psd given by $\varphi_{XX}(\omega)=\frac{16}{16+w2}$ and $\varphi_{YY}(\omega)=\frac{16}{16+w2}$. Let another stationary random process be U(t)=X(t)+Y(t). assume that X(t) and Y(t) are uncorrelated with zero mean; then $\varphi_{UU}(\omega)$=

a) $\Phi_{XX}(\omega)+\Phi_{XY}(-\omega)+\Phi_{YX}(-\omega)+\Phi_{YY}(\omega)$

b) $\Phi_{XX}(-\omega)+\Phi_{XY}(-\omega)+\Phi_{YX}(-\omega)+\Phi_{YY}(-\omega)$

c) $\Phi_{XX}(\omega)+\Phi_{XY}(\omega)+\Phi_{YX}(\omega)+\Phi_{YY}(\omega)$

d) $\Phi_{XX}(-\omega)+\Phi_{XY}(-\omega)+\Phi_{YX}(-\omega)+\Phi_{YY}(-\omega)$

Section-B:

1) Define the following systems

i) Linear System ii) Causal System iii) Stable System

2) Derive the relation for Time-Invariant System Transfer Function.

3) Derive the expression for the following

a) System Response

b) Mean Value of System Response

c) Mean Square Value of System Response & with White Noise as input

d) Auto-Correlation of System Response and with White Noise as input

e) Cross-Correlation of System Response and with White Noise as input

4) Derive the relation between psds of input and output random process of an LTI System

5) A random process X(t) is applied to a network with impulse response $h(t)=e^{-bt}u(t)$, where b>0 is constant. The cross correlation X(t) with the output Y(t) is known to have the form $R_{XY}(\tau)=\tau e^{-b\tau}u(\tau)$. Find autocorrelation of Y(t).

6) A random process X(t) whose mean value is 2 and autocorrelation function is $R_{XX}(\tau)=4e^{-2|\tau|}$ is applied to a system whose transfer function is $\frac{1}{2+j\omega}$. Find the mean value, autocorrelation, power density spectrum and average power of the output signal Y(t).

7) Two systems have transfer functions $H_1(\omega)$ and $H_2(\omega)$.

a) Show that transfer function $H(\omega)$ of the cascade of the two i.e. output of first feeds the input of the second system is $H(\omega)= H_1(\omega) H_2(\omega)$

b) For a cascade of N systems with transfer functions $H_n(\omega)$; n=1,2,……N.

Show that $H(\omega)=\prod_{n=1}^{N} Hn(\omega)$

8) The autocorrelation of a WSS random process X(t) is given by $R_{XX}(\tau)=A\cos(\omega_0\tau)$; where A and ω_0 are constants. Find PSD.

9) Let jointly WSS processes $X_1(t)$ and $X_2(t)$ cause responses $Y_1(t)$ and $Y_2(t)$ respectively; from a linear time-invariant system with impulse responses h(t). if the sum $X(t)=X_1(t)+X_2(t)$ is applied the response is Y(t). Find a) E[Y(t)]; b) $R_{YY}(t, t+\tau)$ in terms of h(t) and characteristics of $X_1(t)$ and $X_2(t)$

10) Let $X_1(t)$ and $X_2(t)$ are WSS processes. Show that

a) $\varphi_{YY}(\omega)=|H(\omega)^2|[\varphi_{x1x1}(\omega) + \varphi_{x1x2}(\omega) + \varphi_{x2x1}(\omega) + \varphi_{x2x2}(\omega)]$

b) if $X_1(t)$ and $X_2(t)$ are statistically independent; then

$\varphi_{YY}(\omega)=|H(\omega)^2|[\varphi_{x1x1}(\omega) + \varphi_{x2x2}(\omega) + 4\pi\overline{X1}\,\overline{X2}\delta(\omega)]$

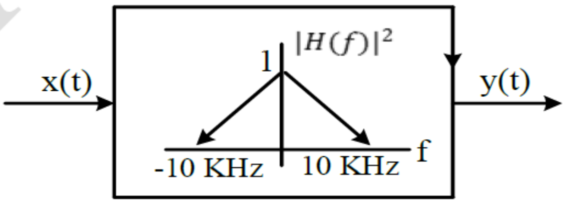
11) Consider a WSS random process X(t) with PSD as $\varphi_{XX}(\omega)$. Another random process is given by Y(t) = X(t+T)+ X(t-T); where T is a constant. Find the PSD of Y(t) ie. $\varphi_{YY}(\omega)$

12) A stationary random signal X(t) has an autocorrelation function $R_{XX}(\tau)=10e^{-|\tau|}$. It is added to a white noise for which $N_o/2 = 10^{-3}$ and the sum is applied to a filter having a transfer function $H(\omega)=2/(1+j\omega)^3$. Find i) power spectrum and ii) average power in the output signal.

Section-C:

Previous GATE Questions:

8. A white noise process X(t) with two-sided power spectral density $1 \times 10^{-10}\ W/Hz$ is input to a filter whose magnitude squared response is shown below.

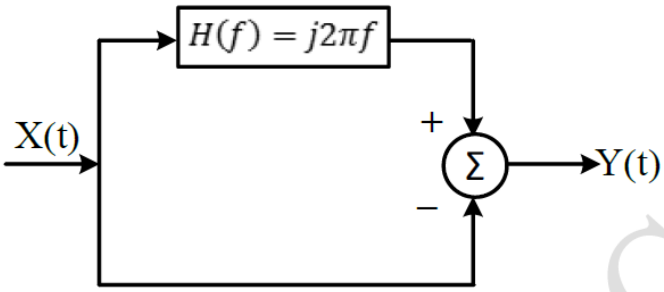


The power of the output process y (t) is given by

(a) $5 \times 10^{-7} W$
(b) $1 \times 10^{-6} W$
(c) $2 \times 10^{-6} W$
(d) $1 \times 10^{-5} W$

[GATE 2009: 1 Mark]

10. X (t) is a stationary random process with autocorrelation function $R_X(\tau) = exp(-\pi\tau^2)$ this process is passed through the system below. The power spectral density of the output process Y(t) is



(a) $(4\pi^2 f^2 + 1)\ exp(-\pi f^2)$
(b) $(4\pi^2 f^2 - 1)\ exp(-\pi f^2)$
(c) $(4\pi^2 f^2 + 1)\ exp(-\pi f)$
(d) $(4\pi^2 f^2 - 1)\ exp(-\pi f)$

[GATE 2011: 2 Marks]